4.1 The unification of electromagnetic and weak interactions

To incorporate electromagnetic interaction we define a weak hypercharge Y and require that the Gell-Mann-Nishijima relation between Y, the 3. component of the weak isospin I_3 and the electric charge Q, holds

$$Q = I_3 + \frac{1}{2}Y$$

We require invariance under the group product $SU(2)_L \times U(1)_Y$

$$\Psi_L(x) \to \Psi'_L(x) = \exp[i(\tau^a \phi^a(x) + Y_L \alpha(x))]\Psi_L(x)$$

$$e_R(x) \to e'_R(x) = \exp[iY_R \alpha(x)]e_R(x)$$

with $\tau^a, a = 1,2,3$ and $Y_{L,R}$, respectively, being the generators of the $SU(2)_L$ and $U(1)_Y$ symmetry groups. This gauge principle leads to a theory with four massless vector particles: the non-abelian SU(2) gauge fields, W^a_μ, and the abelian $U(1)_Y$ gauge field, B_μ, with the kinetic part

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

and the field strength tensor

$$B_{\mu\nu} = \partial\mu B_\nu - \partial\nu B_\mu$$

The gauge fields are introduced via *minimal substitution*, using the gauge covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + ig\frac{\tau^a}{2}W_\mu^a + ig'\frac{Y}{2}B_\mu$$

with the $U(1)_Y$ gauge field transforming as

$$B_\mu \to B_\mu - \frac{1}{g'}\partial_\mu \alpha(x)$$

As a result, the part of the interaction Lagrangian $\mathcal{L}_{I,lep}$ involving neutral gauge bosons reads:

$$\mathcal{L}_{I,lep} = -\frac{1}{2}\left\{\bar{\nu}_{e,L}\gamma^\mu(gW_\mu^3 + g'Y_L B_\mu)\nu_{e,L} + \right.$$

$$\left. +\bar{e}_L\gamma^\mu(-gW_\mu^3 + g'Y_L B_\mu)e_L + \bar{e}_R\gamma^\mu(g'Y_R B_\mu)e_R\right\}$$

Now we can make the following identifications ($Y_L = -1$):

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gW_\mu^3 - g'B_\mu)$$

being the Z boson that couples to the weak neutral current (NC)

$$J^\mu_{NC} = \frac{1}{2}\bar{\nu}_{e,L}\gamma^\mu\nu_{e,L} - \frac{1}{2}\bar{e}_L\gamma^\mu e_L - \sin^2\theta_w J^\mu_{em}$$

whereas the to Z_μ orthogonal field is the photon field

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g' W^3_\mu + g B_\mu)$$

that couples to the conserved electromagnetic current

$$J^\mu_{em} = -\bar{e}_L\gamma^\mu e_L - \bar{e}_R\gamma^\mu e_R = -\bar{\Psi}_e\gamma^\mu\Psi_e$$

Here we have introduced the weak mixing angle θ_w as

$$\sin\theta_w = \frac{g'}{\sqrt{g^2 + g'^2}} \quad \cos\theta_w = \frac{g}{\sqrt{g^2 + g'^2}}$$

and it is obvious that the transition from gauge fields to physical fields

$$A_\mu = \sin\theta_w W^3_\mu + \cos\theta_w B_\mu$$

$$Z_\mu = \cos\theta_w W^3_\mu - \sin\theta_w B_\mu$$

just represents a change of basis in fields. Moreover, we have to set

$$Y_R = -2 \text{ and } \frac{gg'}{\sqrt{g^2 + g'^2}} = e$$

to obtain the QED interaction Lagrangian. As a result we find the

Pattern of unification: $e = g \sin\theta_w = g' \cos\theta_w$

We have replaced three coupling constants by two, the electric charge and the weak mixing angle, where the latter is a measure of the admixture of the electromagnetic currrent to the weak neutral current that couples to the Z boson.