Crossing symmetry

The S matrix element for any process involving a particle with momentum p in the initial state is equal to the S matrix element for an otherwise identical process but with the anti-particle with momentum $k = -p$ in the final state:

$$\mathcal{M}(a(p) + \ldots \to \ldots) = \mathcal{M}(\ldots \to \ldots + \bar{a}(k = -p))$$

For instance, from the matrix element for the process $e^- e^+ \to \mu^- \mu^+$ we can easily obtain the one for $e^- \mu^- \to e^- \mu^-$ by crossing μ^+, e^+.

Interaction in the presence of an external electromagnetic field

So far we considered the interactions taking place in the vacuum. The Feynman-rules can be also applied to calculating transition probabilities in the presence of an external electromagnetic field. For example, electron scattering from an external electromagnetic field, $e^-(p) \to e^-(p')$ reads:

$$2\pi i \mathcal{M} \delta(p^{0'} - p^0) = \tilde{A}_\mu^{ext.}(p' - p)\bar{u}(p')(-ie\gamma^\mu)u(p)$$

where $\tilde{A}_\mu^{ext.}$ denotes the Fourier transform of the external electromagnetic field.

Infrared (IR) singularities

The radiation of massless particles off on-shell particles results into IR singularities. For example, the transition amplitude for photon radiation off electrons scattered in an external electromagnetic field contains terms of the form:

$$\frac{1}{(k+p)^2 - m_e^2} = \frac{1}{2k^0 p^0 (1 - \beta_e \cos\theta)}$$

which diverges for the photon energy $k^0 \to 0$ (*soft* IR singularity) and $\beta_e \cos\theta \to 1$ (*collinear* IR singularity).

These IR divergences exactly cancel in the cross section order-by-order in perturbation theory when also including the corresponding virtual photon emission diagrams.

Ultraviolet (UV) singularities

Radiative corrections, i.e. contributions to the transition amplitude beyond the Born approximation, can also develop UV divergences, i.e. loop integrals that diverge for large loop momenta, $k \to \infty$ (see, e.g., the example in the lecture notes of March 5).

Renormalization:

Removal of UV divergences order-by-order in perturbation theory by the introduction of a finite number of counterterms in the Lagrangian which is equivalent to a redefinition of the parameters of the theory, i.e. couplings, masses and fields. A renormalization procedure must obey the fundamental principles underlying a QFT.

In renormalizable Quantum Field Theories

- the degree of divergence of any Feynman-graph must be a function of only the external legs, i.e. it must remain constant when adding more internal loops,
- must only contain dimensionless coupling constants,
- the number of classes of divergent n-point diagrams must be finite,
- the divergences must cancel against the divergences contained in the "bare" parameters.

QED is a renormalizable QFT

4. The Glashow-Weinberg-Salam model of electroweak interactions

When constructing the Langrangian for the electroweak interaction among leptons and quarks, we have to keep in mind the phenomenology of the weak interaction, i.e. what we already have learned from experiment:

- At low energies, weak interaction processes are well described by a four-fermion point interactions, e.g. for β decay:

$$\mathcal{L} \sim \frac{G_F}{\sqrt{2}} J_\mu^{had}(x) J^{\mu,lept}(x)$$

- Parity violation $\Rightarrow V - A$ structure of the weak interaction:

$$J_\mu^{lept}(x) = \bar{\Psi}_e(x)\gamma_\mu(1-\gamma_5)\Psi_{\nu_e}(x)$$

- Universality: all weak currents couple with the same strength.

Because of the $V - A$ structure of the weak interaction, only left-handed electrons, $e_L = 1/2(1-\gamma_5)\Psi_e$, and neutrinos, $\nu_{e,L} = 1/2(1-\gamma_5)\Psi_{\nu_e}$, occur in β decay. With respect to the weak interaction e_L and $\nu_{e,L}$ are not distinguishable when neglecting the small mass difference. Thus, they can be viewed as manifestations of one weak state Ψ_L

$$\Psi_L(x) = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}$$

and the free theory, $\mathcal{L}_0 = \bar{\Psi}_L(x)(i\not{\partial})\Psi_L(x)$, is invariant under transformation of the global $SU(2)$ symmetry group of the weak isospin $I^a = \tau^a/2$:

$$\Psi_L(x) \to \Psi'_L(x) = U\Psi_L(x)$$

with $U = \exp[i/2\tau^a\alpha^a] = 1 + i/2\tau^a\delta\alpha^a, a = 1, 2, 3.$

But global symmetries are not considered "fundamental", since they contradict the idea of local gauge field theory.

In the construction of $\mathcal{L}$ of the weak interaction we identify the requirement to be invariant under a local $SU(2)$ transformation in the weak isospin space as the fundamental gauge principle.

We consider left-handed fermions to be grouped in the fundamental representation of the $SU(2)_L$ symmetry group of the weak isospin I_w. Right-handed electrons, $e_R = 1/2(1+\gamma_5)\Psi_{\nu_e}$, form a $SU(2)_L$ singlet. We require the Lagrangian to be invariant under local $SU(2)_L$ transformations

$$\Psi_L(x) \to \Psi'_L(x) = U(x)\Psi_L(x)$$

$$e_R(x) \to e_R(x)$$

with

$$U(x) = \exp[\frac{i}{2}\tau^a\alpha^a(x)] = 1 + \frac{i}{2}\tau^a\delta\alpha^a(x)$$

The generators of the group $\tau^a, a = 1, 2, 3$ are the Pauli spin matrices which obey the following algebra

$$[\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c$$

where ϵ^{abc} are the $SU(2)_L$ structure constants.

As in the case of QED, gauge invariance requires the introduction of vector fields. We consider the gauge covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + igW_\mu$$

with $W_\mu = 1/2\tau^a W_\mu^a, a = 1, 2, 3$ which introduces an interaction of the left-handed fermions with the $SU(2)_L$ vector fields W_μ^a. The kinetic term for the weak gauge fields reads

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a}$$

with the weak field strength tensor

$$F^a_{\mu\nu} = \partial\mu W^a_\nu - \partial\nu W^a_\mu - g\epsilon^{abc} W^b_\mu W^c_\nu$$

Together with a prescription how these gauge fields transform

$$W_\mu \to U(x) W_\mu U^\dagger(x) - \frac{i}{g} U \partial_\mu U^\dagger$$

we find that the Langrangian describing weak interactions among leptons

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

with the free Lagrangian

$$\mathcal{L}_0 = \bar{\Psi}_L(x)(i\gamma^\mu \partial_\mu)\Psi_L(x) + \bar{e}_R(x) i\gamma^\mu \partial_\mu e_R(x)$$

$$+\frac{1}{2} W^a_\mu (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) W^a_\nu$$

and the interaction part

$$\mathcal{L}_I = \bar{\Psi}_L(x)(-g\gamma^\mu W_\mu)\Psi_L(x) + g\epsilon^{abc}(\partial^\mu W^{\nu,a})W_\mu^b W_\nu^c +$$

$$-\frac{1}{4}g^2(W^{\mu,a}W^{\nu,b}W_\mu^a W_\nu^b - W^{\mu,a}W^{\nu,a}W_\mu^b W_\nu^b)$$

is invariant under local $SU(2)_L$ transformations.

Let's have a closer look at the part of $\mathcal{L}_I$ that describes interactions of leptons with the gauge fields W_μ^a:

$$\mathcal{L}_{I,lep} = \bar{\Psi}_L(x)(-\frac{g}{2}\gamma^\mu \tau^a W_\mu^a)\Psi_L(x) =$$

$$= \bar{\Psi}_L(x)(-\frac{g}{2}\gamma^\mu)\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\Psi_L(x) =$$

$$= -\frac{g}{2}\left\{\bar{\nu}_{e,L}\gamma^\mu W_\mu^3 \nu_{e,L} + \sqrt{2}\bar{\nu}_{e,L}\gamma^\mu W_\mu^+ e_L + \right.$$

$$\left. + \sqrt{2}\bar{e}_L\gamma^\mu W_\mu^- \nu_{e_L} - \bar{e}_L\gamma^\mu W_\mu^3 e_L\right\}$$

with

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$$

The vector field W_μ^3 cannot be identified with the photon, since it also couples to left-handed neutrinos.

Thus, as a consequence of the gauge principle, a new form of weak interaction arises:

The GWS model predicts the existence of a neutral weakly interacting boson, that mediates a weak interaction (neutral currents).

However, we still need to include the electromagnetic interaction and the fact that electrons and the weak gauge bosons are massive.