For the calculation of Green's function for the full theory described by the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I + \mathcal{L}_{gf}$, where $\mathcal{L}_I$ describes the interaction, we rely on a perturbative expansion in terms of powers of $S_I = \int d^4x \mathcal{L}_I$.

Let's consider the Klein-Gordon field theory again with interaction. We can rewrite the path integral of interacting fields as follows

$$Z[J] = \int \mathcal{D}[\Phi] \exp[i \int d^4x(\mathcal{L}_0 + \mathcal{L}_I(\Phi) + J(x)\Phi(x))] =$$

$$= \int \mathcal{D}[\Phi] \exp[i \int d^4x \mathcal{L}_I(\Phi \to \frac{1}{i}\frac{\delta}{\delta J})] \exp[i \int d^4x(\mathcal{L}_0 + J(x)\Phi(x))] =$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}(iS_I(\frac{1}{i}\frac{\delta}{\delta J}))^m Z_0[J]$$

The n-point Green's function can be obtained order-by-order in perturbation theory from the generating functional $Z(J)$ as follows

$$G^{(n)}(x_1,\dots,x_n) = (\frac{1}{i})^n \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_n)} \left. \frac{Z[J]}{Z[0]} \right|_{J=0} =$$

$$= (\frac{1}{i})^n \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_n)} \sum_{m=0}^{\infty} \frac{1}{m!} (iS_I(\frac{1}{i}\frac{\delta}{\delta J}))^m$$

$$(\exp[-\frac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)])_{J=0} \frac{Z_0[0]}{Z[0]}$$

with $Z_0[0] = \int \mathcal{D}[\Phi] \exp[i \int d^4x \mathcal{L}_0]$.

An example: $\mathcal{L}_I = -\frac{\lambda}{4!}(\Phi(x))^4$

At order λ^0 ($m = 0$) in perturbation theory we recover again the free field theory.

At order λ^1 $(m = 1)$ in perturbation theory we find

$$G^{(n)}(x_1, \dots, x_n) =$$

$$= (\frac{1}{i})^n \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_n)} (1 - i\frac{\lambda}{4!} \int d^4z (\frac{1}{i} \frac{\delta}{\delta J})^4)$$

$$(\exp[-\frac{i}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y)])_{J=0} \frac{Z_0[0]}{Z[0]}$$

At this order in λ there are two contributions that are "connected", a two-point function

$$G^{(2)}(x_1, x_2) = i\Delta(x_1 - x_2) + \frac{-i\lambda}{4!} 6 \int d^4z \; i\Delta(0) i\Delta(z - x_1) i\Delta(z - x_2)$$

where the second term corresponds to a loop contribution, and a four-point function

$$G^{(4)}(x_1, x_2, x_3, x_4) =$$

$$-i\frac{\lambda}{4!} \int d^4z i\Delta(z - x_1) i\Delta(z - x_2) i\Delta(z - x_3) i\Delta(z - x_4)$$

which is called a tree-level contribution.

We can graphically represent the Green's functions as a sum of Feynman-diagrams, when we draw a line for each propagator $i\Delta$ and a dot for the interaction vertices.

The result can be expressed in terms of Feynman-rules that are prescribing how to construct Feynman diagrams and how to write down the corresponding analytic expressions order-by-order in perturbation theory.

Feynman-rules allow for a quick derivation of the Green's function of a field theory without having to go through the whole formalism again and again.

Loop diagrams are genuine QFT effects.

Unfortunately, they exhibit divergences which are an artifact of perturbation theory.

For instance, the one-loop contribution shown above can be rewritten in terms of the Fourier transforms of Δ, denoted by $\tilde{\Delta}$, as follows

$$G^{(2)}(x_1, x_2) = -i\frac{\lambda}{4!}6\int \frac{d^4k}{(2\pi)^4} i\tilde{\Delta}(k) \int d^4z \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}$$

$$i\tilde{\Delta}(k_1) i\tilde{\Delta}(k_2) e^{-i(k_1(z-x_1)+k_2(z-x_2))} =$$

$$= -i\frac{\lambda}{4!}6\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} i\tilde{\Delta}(k_1) i\tilde{\Delta}(k_2) e^{i(k_1x_1+k_2x_2)}\delta^4(k_1+k_2)\int \frac{d^4k}{(2\pi)^4} i\tilde{\Delta}(k)$$

One integration is not a Fourier transformation

$$\int \frac{d^4k}{(2\pi)^4} i\tilde{\Delta}(k) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}$$

and leads to a Ultra-Violet (UV) divergence, i.e. it is divergent for $k \to \infty$.

In renormalizable field theories, UV divergences can be removed by a suitable renormalization procedure.

So far, we only dealt with observables of the type of number operators, i.e. $f(k)a^\dagger(\vec{k})a(\vec{k})$ which "measure" for instance energy and momentum of sums of free one-particle states.

But what we are really interested in, and what is experimentally studied in accelerator experiments, is the dynamics of interacting elementary particles, i.e. *scattering, annihilation and creation of particles*.

Surprisingly, we can describe such complex processes based on quite a strong idealization:

All processes are considered to be transitions of one basis of the Hilbert space of free particles to another one, i.e.

$$\Phi_{in} \rightarrow \Phi_{out}$$

so that there exists an unitary operator S, the scattering matrix, as follows

$$\Phi_{in} = S^{-1}\Phi_{out} \quad , S^\dagger S = SS^\dagger = 1$$

$\Phi_{in}(\Phi_{out})$ are a collection of free asymptotic states at $t \rightarrow -\infty(+\infty)$.

The matrix elements of S

$$S_{fi} =< p_1 \ldots p_n|S|q_1 \ldots q_m >= \delta_{if} + i\delta^4(\sum_{i=1}^{n} q_i - \sum_{j=1}^{m} p_j) < p_1 \ldots p_n|T|q_1 \ldots q_m >$$

describe the probability for a transition of an initial state consisting of n particles with momenta $q_1, \ldots q_n$ to a final state consisting of m particles with momenta $p_1, \ldots p_m$. The non-interacting part is extracted and the transition matrix T is introduced.

The Green's functions can be used to determine the matrix elements of S. A general structure of the S operator can be obtained solely based on general principles, i.e. the requirements of Lorentz invariance, causality and locality (see Bogoliubov and Shirkov in *Theory of interacting fields*) and can be evaluated order-by-order in perturbation theory as follows

$$S = Te^{i\int d^4x \mathcal{L}_I} = 1 + \sum_n \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n T(\mathcal{L}_I(\phi(x_1)), \ldots \mathcal{L}_I(\phi(x_n)))$$

i.e. the scattering matrix is completely determined by the interaction Lagrangian.

For the evaluation of the S matrix elements between different states, we have to find a way how to deal with the asymptotic fields.

Using the Lehman-Symanzik-Zimmermann (LSZ) formalism (see Bjorken and Drell), the asymptotic fields $\Phi_{in,out}$ can be removed from the S matrix elements in favor of the interacting fields, Φ, when assuming the weak asymptotic limit. For Klein-Gordon fields this reads

$$\Phi(x) \to \sqrt{z}\Phi_{in}(x) \quad x_0 \to -\infty$$

$$\Phi(x) \to \sqrt{z}\Phi_{out}(x) \quad x_0 \to \infty$$

where this limit is only applied to the matrix elements of Φ. The S matrix elements (here given for a Klein-Gordon field theory) can then be expressed in terms of the Green's functions as follows

$$< p_1 \dots p_n|S|q_1 \dots q_m >= (\frac{i}{\sqrt{z}})^{n+m} \lim \int \prod_{k=1}^{m} d^4x_k(\partial_x^2 + m^2)$$

$$G^{(n+m)}(y_1, \dots y_n; x_1 \dots x_m) \prod_{j=1}^{n} d^4y_j(\partial_y^2 + m^2) e^{i(\sum_j p_j y_j - \sum_k q_k x_k)} =$$

$$= (\frac{-i}{\sqrt{z}})^{n+l} \lim \Pi(p_k^2 - m^2)\Pi(q_j^2 - m^2)\tilde{G}^{(n+l)}(p_1, \dots p_n; q_1 \dots q_m)$$

$\lim : q_i^2 \to m^2, p_i^2 \to m^2, q_i^0 > 0, p_i^0 > 0.$

Thus, the S matrix elements are constructed from the Green's functions by replacing in configuration space the propagators of the external fields by wave functions, e.g. for Klein-Gordon fields: $< k|\Phi(x)|0 > \propto \exp[-ikx]$, describing real (on-mass shell) free external particles.