
Note: For the calculation of Green's function in more complex theories, e.g. when including interaction $S = S_0 + S_I$, we rely on a perturbative expansion in terms of powers of S_I (powers of the coupling constant)

$$e^{iS} = e^{iS_0} \sum_{n=0}^{\infty} \frac{1}{n!} (iS_I)^n$$

Now consider the functional integral for the free electromagnetic field

$$\begin{aligned} & \int \mathcal{D}[A_\mu] \exp\left[i \int dx \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu(x) A_\mu(x)\right)\right] = \\ & = \int \mathcal{D}[A_\mu] \exp\left[i \int dx \left(\frac{1}{2} A_\mu(x) (g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) A_\nu(x) + J_\mu(x) A^\mu(x)\right)\right] \end{aligned}$$

However, if we try to solve this with the help of the Gauss integration, we find that this does not work, since there is no inverse of the operator $K^{\mu\nu} = g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu$ (it has eigenvalues 0). $K^{\mu\nu} = P^{\mu\nu} \partial^2$ where P is a projection operator, i.e. its square is equal to itself

$$P^{\mu\nu} P_{\nu\rho} = P_\rho^\mu$$

It projects out longitudinal degrees of freedom of the electromagnetic field.

That we cannot “naively” construct a propagator for the gauge field is a property of any gauge field theory.

The functional integral is not well-defined, since we redundantly integrate over a continuous infinity of field configurations that are physically equivalent.

Gauge invariance leaves us the freedom to choose $A_\mu(x) = \partial_\mu\alpha(x)$, i.e. those fields which are gauge-equivalent to $A_\mu = 0$. Then the action S vanishes and the functional integral is divergent.

The solution is that we have to make sure that we only integrate over those field configurations that are not connected by gauge invariance.

Faddeev and Popov accomplished this by means of a trick where a 1 is inserted in the generating functional so that it is restricted to only those field configurations that are physically inequivalent (i.e. not connected by a gauge transformation).

First we define a gauge condition $F[A_\mu] = 0$ which fixes or “defines” the gauge. F is an arbitrary functional of the gauge fields A_μ .

An example: Lorentz or Landau gauge

$$F[A_\mu] = \partial^\mu A_\mu = 0 \Rightarrow F[A_\mu^g] = \partial^\mu A_\mu - \partial^2 \alpha(x) = 0 \text{ for } \partial^2 \alpha(x) = 0$$

A_μ^g denotes the result when a gauge transformation is applied to A_μ .

We could try to constrain the functional integral to only those field configurations with $F[A_\mu] = 0$ by inserting this condition with a functional δ -function. However, we have to be careful, since simply inserting $\delta[F]$ in the $\mathcal{D}[A_\mu]$ integral changes the measure of the integration. Remember, that when a function $f(x)$ has a zero at $x = a$ the choice $\delta(x - a)$ differs from the choice $\delta(f(x))$ by

$$\delta(f(x)) = \delta(x - a) \frac{1}{|f'(x)|}$$

Faddeev and Popov (1967) suggested to insert the following instead

$$1 = \int \mathcal{D}[\alpha(x)] \delta[F[A_\mu^g]] \det \left(\frac{\delta[F[A_\mu^g]]}{\delta \alpha} \right)_{F=0}$$

which we know has the right measure.

We then obtain for the functional integral (we can replace A_μ^g by A_μ since the functional integral is gauge invariant)

$$Z[J] = \int \mathcal{D}[\alpha(x)] \int \mathcal{D}[A_\mu] \delta[F[A_\mu]] \det \left(\frac{\delta[F[A_\mu]]}{\delta\alpha} \right)_{\alpha=0} \exp[iS[A_\mu]]$$

For practical purposes we would like to rewrite the delta function. For that we consider a class of gauge conditions $F[A_\mu] - C(x) = 0$. This does not change the Fadeev-Popov determinant. Since being gauge invariant, the above functional integral is independent of $C(x)$ and we can integrate functionally over an arbitrary weight functional $G[C]$ as follows (N denotes some constants)

$$\begin{aligned} Z[J] &= N \int \mathcal{D}[\alpha(x)] \int \mathcal{D}[A_\mu] \det \left(\frac{\delta[F[A_\mu]]}{\delta\alpha} \right)_{\alpha=0} \exp[iS[A_\mu]] \\ &\quad \int \mathcal{D}[C] \delta[F[A_\mu - C(x)]] G[C] = \\ &= \int \mathcal{D}[\alpha(x)] \int \mathcal{D}[A_\mu] \det \left(\frac{\delta[F[A_\mu]]}{\delta\alpha} \right)_{\alpha=0} \exp[iS[A_\mu]] G[F[A_\mu]] \end{aligned}$$

If we choose for $G[C]$

$$G[C(x)] = \exp\left[-\frac{i}{2\xi} \int d^4x C(x)^2\right]$$

the delta function $\delta[F]$ becomes equivalent to adding a term to the Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L} - \frac{i}{2\xi} (F[A_\mu])^2$$

ξ is an arbitrary real number.

Of course, being gauge invariant the functional integral does not depend on ξ and thus also the physical results obtained from it are independent of ξ .

Finally, let's calculate the propagator for a free electromagnetic field using as gauge condition $F[A_\mu] = \partial^\mu A_\mu$. With this choice the Fadeev-Popov determinant ($=\partial^2$) does not depend on the fields and the path integral reads

$$Z[J] = N \int \mathcal{D}[A_\mu] \exp[i \int d^4x (\frac{1}{2} A_\mu(x) (g^{\mu\nu} \partial^2 - (1 - \frac{1}{\xi}) \partial^\mu \partial^\nu) A_\nu(x) + J_\mu(x) A^\mu(x))]]$$

and we find for the generating functional

$$\frac{Z[J]}{Z[0]} = \exp[\frac{1}{2} i \int d^4x d^4y J_\mu(x) \Delta^{\mu\nu}(x-y) J_\nu(y)]$$

with

$$[\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu (1 - \frac{1}{\xi}) - i\epsilon] \Delta_\rho^\nu(x-y) = -\delta_{\mu\rho} \delta^4(x-y)$$

with the solution

$$\Delta_{\mu\nu}(x - y) = \int \frac{d^4 k}{(2\pi)^4} \left[g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right] \frac{1}{k^2 + i\epsilon} e^{-ik(x-y)}$$

This is the Feynman propagator in configuration space,

$G_F^{\mu\nu}(x, y) = -i\Delta^{\mu\nu}(x - y)$, for the free electromagnetic field describing the movement of a (virtual) photon with momentum k in space-time.

Now we can write down our 2. Feynman rule by which we will construct Green's functions:

For each internal photon line, associate a propagator given by

$$\text{•} \text{---} \text{•} : \frac{-i}{k^2 + i\epsilon} \left[g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right]$$

Feynman gauge: $\xi = 1$, Landau gauge: $\xi = 0$.