
Quantization of the photon field

Because of gauge invariance there are complications when trying to quantize the photon field “naively”, i.e. analogous to the Klein-Gordon field. When following the procedure for the Klein-Gordon field, one finds physical states with negative norm and unwanted, unphysical polarization degrees of freedom for the photon, and one finds that one cannot construct a photon propagator for the theory.

The difference is that we have to implement in the quantization procedure an additional constraint, i.e. that all fields A_μ which only differ by a gauge transformation are equivalent. This is achieved by *fixing the gauge*.

Quantization by using Feynman’s path integral formalism :

The path integral quantization method involves classical fields only and the theory is formulated directly in terms of Green’s functions given by path integrals.

In Feynman’s path integral representation of QFT, the Green’s functions of a theory are given as functional derivatives of the generating functional

$$Z\{j\} \sim \int \mathcal{D}[\Phi] \exp[i \int d^4x (\mathcal{L}(\Phi(x)) + j(x)\Phi(x))]$$

where $j(x)$ are sources of the Φ fields.

The n -point Green's functions are obtained by taking the functional derivatives with respect to J of the generating functional

$$G^{(n)}(x_1, \dots, x_n) = \left(\frac{1}{i}\right)^n \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_n)} \frac{Z[J]}{Z[0]} \Bigg|_{J=0}$$

First a few facts about functionals:

A functional $F[y(x)]$ is a function that maps functions $y(x)$ to numbers. A functional can be integrated over a set of functions $y(x)$ and the measure is conventionally written as $\mathcal{D}[y]$. It can also be differentiated with respect to $y(x)$: $\delta F/\delta y(x)$ and the rules for differentiating functions can be readily generalized to functionals (in four dimensions):

$$\frac{\delta y(x')}{\delta y(x)} = \delta^4(x - x')$$

or

$$\frac{\delta}{\delta y(x)} \int d^4 x' y(x') z(x') = z(x)$$

$$\frac{\delta}{\delta y(x)} (F[y]G[y]) = \left(\frac{\delta}{\delta y(x)} F[y]\right)G[y] + F[y]\left(\frac{\delta}{\delta y(x)} G[y]\right)$$

The differential measure is translation invariant so that

$$\begin{aligned} \int \mathcal{D}[y] F[y - y'] \exp[-(y - y', A(y - y'))] &= \\ &= \int \mathcal{D}[y] F[y] \exp[-(y, Ay)] \end{aligned}$$

where we used the abbreviation (A is a matrix)

$$(y, Ay) = \int dx \int dx' y(x) A(x, x') y(x')$$

Functional integrals of this form can be evaluated by using a generalization of the Gauss integral, if $\det A > 0$

$$\int \mathcal{D}[y] \exp[-(y, Ay)] = (\det A)^{-1/2}$$

where the functional determinant $\det A$ is given by the product of the eigenvalues of the operator $A(x, x')$.

An example:

Correlation functions for the Klein-Gordon field theory

Consider a free Klein-Gordon field theory described by

$$\mathcal{L}_0 = \frac{1}{2}(\partial^\mu \Phi(x)\partial_\mu \Phi(x) - m^2 \Phi^2)$$

$$Z[J] = \int \mathcal{D}[\Phi] \exp[i \int d^4x (\mathcal{L}_0 + J(x)\Phi(x))]$$

Let's rewrite the integrand as follows:

$$\int d^4x (\mathcal{L}_0 + J(x)\Phi(x)) = \frac{1}{2} \int d^4x [\Phi(x) + \int d^4y \Delta(x-y)J(y)](-\partial^2 - m^2 + i\epsilon)$$

$$[\Phi(x) + \int d^4y \Delta(x-y)J(y)] - \frac{1}{2} \int d^4x d^4y J(x)\Delta(x-y)J(y)$$

with

$$(-\partial^2 - m^2 + i\epsilon)\Delta(x-y) = \delta^4(x-y)$$

and the damping factor $1/2i\epsilon\Phi^2$ has been added so that the functional integral is well defined and to ensure the correct boundary conditions for the Green's function.

A shift in the fields $\Phi \rightarrow \Phi' = \Phi(x) + \int d^4y \Delta(x-y)J(y)$ leaves the functional integral unchanged (Jacobian=1) so that

$$\begin{aligned} Z[J] &= \int \mathcal{D}[\Phi] \exp[i \int d^4x \mathcal{L}_0] \exp[-\frac{i}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y)] = \\ &= Z[0] \exp[-\frac{i}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y)] \end{aligned}$$

The two-point Green's function describing the propagation of a free Klein-Gordon particle is obtained as follows:

$$\begin{aligned} G^{(2)}(x_1, x_2) &= \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \frac{Z[J]}{Z[0]} \Big|_{J=0} = \\ &= \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \exp[-\frac{i}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y)] \Big|_{J=0} = \\ &= \frac{\delta}{\delta J(x_1)} \frac{i}{2} \left\{ \int d^4y \Delta(x_2-y) J(y) + \int d^4x J(x) \Delta(x-x_2) \right\} \frac{Z[J]}{Z[0]} \Big|_{J=0} \\ &= i \Delta(x_1 - x_2) = \langle 0 | T(\Phi(x_1) \Phi(x_2)) | 0 \rangle \end{aligned}$$

The generating functional $Z[J]/Z[0]$ can only be written in the above closed form for theories (=Lagrangians) containing terms at most quadratic in fields.