Now we can write down our first Feynman rule by which we will construct Green's functions (they are usually expressed in momentum space):

For each internal fermion line, associate a propagator given by

$$: \frac{i}{\gamma^\mu p_\mu - m + i\epsilon}$$

The Green's functions describe the physics content of a field theory.

3.4 The QED Lagrangian

After having quantized the free Dirac particle, as a next step towards a QFT of electromagnetic interaction among fermions, we need to couple the spin 1/2 Dirac particle, $\Psi(x)$, to a spin 1 Maxwell field A_μ.

With the help of the electromagnetic field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, where A^μ denotes the electromagnetic four-potential $A^\mu = (V, \vec{A})$ with

$$\vec{E} = -\vec{\nabla} V - \frac{\partial}{\partial_t}\vec{A}\,;\; \vec{B} = \vec{\nabla} \times \vec{A}$$

Maxwell's equations can be expressed compactly in convariant form as follows:

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\epsilon_{\mu\nu\rho\sigma}\partial_\nu F^{\rho\sigma} = 0$$

with the four-current $j^\mu = (\rho, \vec{j})$. An appropriate Lagrangian for the free electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

from which Maxwell's equations in the absence of sources can be derived.

The electromagnetic potential is not uniquely defined:

Any electromagnetic potential $\tilde{A}^\mu = A^\mu + \partial^\mu \Lambda(x)$ describes the same electromagnetic field strength tensor. This freedom is called *gauge invariance*.

The idea of using gauge invariance as a dynamical principle originates from H.Weyl's attempts of finding a common geometric basis for both electromagnetism and gravity by requiring invariance under a space-time dependent change of scale (*Eichinvarianz*, Eich=gauge, standard of calibration).

The requirement of gauge invariance, i.e. invariance of the Dirac Lagrangian under local phase rotations

$$\Psi(x) \to \Psi(x) e^{iq\alpha(x)}$$

$$\mathcal{L} \to \mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - q\bar{\Psi}(x)\gamma^\mu \partial_\mu \alpha(x)\Psi$$

necessitates the introduction of a vector field A_μ (spin 1) which transforms as follows

$$A_\mu \to A_\mu - \partial_\mu \alpha(x)$$

and, following the *minimal coupling prescription*, couples to the Dirac field via the gauge covariant derivative

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + iqA_\mu .$$

The vector field is called a gauge field and the associated particles are called gauge bosons.

After identifying A_μ as the electromagnetic four-potential and q as the electric charge, the kinetic energy of the vector field, which describes the propagation of free photons, is added, and we find the QED Lagrangian

$$\mathcal{L} = \bar{\Psi}(x)(i\gamma^\mu \mathcal{D}_\mu - m)\Psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

which is invariant under local phase rotations. Note that a mass term for the photon $m^2 A^\mu A_\mu$ would violate local gauge invariance.

We obtain the equations of motion from Hamilton's principle of least action: the Euler-Lagrange equation for $\Psi(x)$ yields

$$(i\gamma^\mu \mathcal{D}_\mu - m)\Psi(x) = 0$$

and for A_ν

$$\partial_\mu F^{\mu\nu} = e\bar{\Psi}\gamma^\nu\Psi = j^\nu$$

which is the inhomogenoeous Maxwell equation with the current density given by the conserved Noether vector current.