The requirement of (micro)causality for the free Dirac fields naturally leads to the Pauli exclusion principle for the corresponding particles (=field quanta).

A state vector containing two electrons

$$|k_1, r, k_2, s>= \int d^3k_1 d^3k_2 \sum_{rs} f_r(\vec{k}_1) f_s(\vec{k}_2) a_r^\dagger(\vec{k}_1) a_s^\dagger(\vec{k}_2)|0>$$

vanishes, if the electrons are in the same quantum state because of the postulated anticommutation relations. This is an example for the *spin-statistics theorem*: Spin 1/2 particles are fermions, i.e. they have to be quantized according to Fermi-Dirac spin statistics by using anticommutation relations.

Moreover,

$$\int d^3k_1 d^3k_2 \sum_{rs} f_r(\vec{k}_1) f_s(\vec{k}_2) a_r^\dagger(\vec{k}_1) a_s^\dagger(\vec{k}_2)|0>=$$

$$= -\int d^3k_1 d^3k_2 \sum_{rs} f_r(\vec{k}_1) f_s(\vec{k}_2) a_s^\dagger(\vec{k}_2) a_r^\dagger(\vec{k}_1)|0>$$

so that

$$|k_1, r, k_2, s>= -|k_2, s, k_1, r>$$

Again, as it was the case for the Klein-Gordon field, we have to abandon the notion that the field operator Ψ describes a single particle wave function - it always contains both, particle and anti-particle.

The Pauli exclusion principle follows from the anticommutation relations and thus the multi-particle state for fermions reads

$$\prod_{i=1}^{N} a_{s_i}^{\dagger}(k_i) \prod_{j=1}^{M} b_{s_j}^{\dagger} |0>$$

with only one particle in any given quantum state.

Fields with integer spin have to be quantized according to Bose-Einstein statistics, i.e. by using commutation relations, and thus the multi-particle state for bosons reads

$$|k_1, \ldots, k_m >= \prod_{i=1}^{m} \frac{(a^{\dagger}(k_i))^{n(k_i)}}{\sqrt{n(k_i)}} |0>$$

where $n(k_i)$ denotes the number of particles in the same quantum state with four-momentum k_i.

To show that the Dirac particle has spin 1/2, we use the Noether theorem and obtain as a consequence of rotational invariance of the Dirac Lagrangian the angular momentum tensor as follows:

$$\mathcal{M}^{\lambda\mu\nu} = i\bar{\Psi}(x)\gamma^\lambda(x^\mu\partial^\nu - x^\nu\partial^\mu - \frac{i}{2}\sigma^{\mu\nu})\Psi(x)$$

with $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]$. The conserved angular momentum tensor then reads

$$M^{\mu\nu} = \int d^3x M^{0\mu\nu} =$$

$$= \int d^3x i\bar{\Psi}(x)\gamma^0(x^\mu\partial^\nu - x^\nu\partial^\mu - \frac{i}{2}\sigma^{\mu\nu})\Psi(x)$$

where the last term arises due to the non-trivial spin. Let's consider a particle at rest ($\vec{k} = 0$) and a rotation around the third component $x^3 = z$. Then only the spin part contributes and we find

$$J_z = M^{03} = \int d^3x i\bar{\Psi}(x) \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \Psi(x)$$

so that

$$J_z a_s^\dagger(0)|0>= \frac{1}{2m}\sum_r (u_r^\dagger(0) \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} u_s(0))a_s^\dagger(0)|0>=$$

$$= \sum_r (\chi_r^\dagger \sigma^3 \chi_s) a_r^\dagger |0>$$

After performing the sum over r by choosing the spinors to be eigenstates of σ^3, we find that for $\chi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the one-particle state is an eigenstate of J_z with eigenvalue $1/2$ and for $\chi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ it is an eigenstate of J_z with eigenvalue $-1/2$, i.e.

$$J_z a_s^\dagger(0)|0>= \pm\frac{1}{2} a_s^\dagger(0)|0>$$

where the upper sign is for $\chi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the lower sign is for $\chi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

This is the result we expect for an electron.

For the positron we find

$$J_z b_s^\dagger(0)|0> = \mp\frac{1}{2} b_s^\dagger(0)|0>$$

where the upper sign is for $\chi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the lower sign is for $\chi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Interpretation of the Dirac field operator $\Psi(x)$

$\Psi(x)$ creates quanta of type b_s with positive energy and annihilates those of type a_s:

$$\Psi(x)|0>=\sum_s\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}v_s(k)e^{ikx}b_s^\dagger|0>=\Psi^{(-)}(x)|0>$$

$\bar{\Psi}(x)$ creates quanta of type a_s with positive energy and annihilates those of type b_s:

$$\bar{\Psi}(x)|0>=\sum_s\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\bar{u}_s(k)e^{ikx}a_s^\dagger|0>=\bar{\Psi}^{(-)}(x)|0>=$$

$$=\overline{\Psi^{(+)}}(x)|0>$$

where $\Psi^{(+)}$ and $\Psi^{(-)}$ denote the solution of the Dirac equation with negative frequency (with positive energy eigenvalue) and positive frequency (with negative energy eigenvalue), respectively.

In QED we will use the Dirac spinor field to describe electrons and positrons. The particles created by $a_s^\dagger$ are electrons with four-momentum k and spin 1/2 and charge +1 (in units of e) and a polarization appropriate to χ_s. The particles created by $b_s^\dagger$ are positrons with four-momentum k and spin 1/2 and charge -1 (in units of the electron charge $e = -|e|$) and a polarization opposite to χ_s. Then, the state $\Psi_a(x)|0>$ contains a positron at position x and the state $\bar{\Psi}_a(x)|0>$ contains an electron at position x, where the polarization corresponds to the spinor component $(a = 1, 2, 3, 4)$ chosen.

How does the Dirac particle propagate in space-time ?

Consider an electron created at $(t, \vec{x})$ and reabsorbed again (=annihilated) into the vacuum at $(t', \vec{x}')$. Then, the corresponding propagation amplitude reads

$$< 0|\Psi_a^{(+)}(x')\bar{\Psi}_b^{(-)}(x)|0> \theta(t'-t)$$

However, in QFT this is equivalent to the creation of a positron at $(t', \vec{x}')$ and its annihilation at $(t, \vec{x})$ with $t > t'$, i.e.

$$< 0|\bar{\Psi}_b^{(+)}(x)\Psi_a^{(-)}(x')|0> \theta(t-t')$$

Thus the propagation amplitude for a free Dirac particle is defined as the superposition of the two amplitudes

$$G_F(x, x') =< 0|\Psi_a^{(+)}(x')\bar{\Psi}_b^{(-)}(x)|0> \theta(t'-t)- < 0|\bar{\Psi}_b^{(+)}(x)\Psi_a^{(-)}(x')|0> \theta(t-t')$$

$$=< 0|T(\Psi_a(x')\bar{\Psi}_b(x))|0 >$$

G_F is called the Feynman propagator. T denotes the chronological ordering of the field operators.

Using the explicit expression for $\Psi(x)$ and the integral representation of the θ function, we find for the propagation amplitude of a free Dirac particle

$$G_F(x'-x) = \int \frac{d^4k}{(2\pi)^4} \frac{i(\gamma^\mu k_\mu + m)}{k^2 - m^2 + i\epsilon} e^{-ik(x'-x)}$$

The term proportional to ϵ denotes how the poles in the complex ω_k plane are (for $\epsilon \to 0$) circumvented, i.e. along which contour the ω_k integral is performed.

The Feynman propagator describes the probability amplitude for a Dirac particle to propagate between the space-time points x and y that is compatible with Lorentz invariance.

G_F solves the differential equation

$$(i\gamma^\mu \partial_\mu - m)G_F(x) = i\delta^4(x)$$

Thus, the Feynman propagator is a Green's function of the Dirac operator $(i\gamma^\mu \partial_\mu - m)$.

Here we make contact between the theory of propagators where scattering amplitudes are written in terms of Green's functions and Quantum Field Theory where everything is expressed in terms of quantized fields, i.e. operators.