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## 3.2 The Dirac equation

The fundamental constituents of matter are fermions (leptons and quarks), i.e. particles with spin  $\frac{1}{2}$ . A scalar field cannot describe a particle with spin. To describe non-relativistic electrons Pauli suggested to use a wavefunction with two components

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$$

where  $|\Psi_{1,2}(x)|^2 d^3x$  is the probability to find an electron with spin in positive(negative) z-direction in the volume  $d^3x$  around the point  $\vec{x}$ . The angular momentum operator is given by  $\vec{J} = \vec{L} + \vec{S}$  ( $\vec{L} = \vec{x} \times \vec{p}$ ,  $\vec{p} = 1/i\vec{\partial}$ ) with the spin angular momentum  $\vec{S} = 1/2\vec{\sigma}$  where  $\sigma_i, i = 1, 2, 3$  are the three Pauli spin matrices. According to Pauli the wave function for a free electron should obey the Schroedinger equation. However, the Schroedinger equation is not relativistic invariant.

Dirac's ansatz to describe a relativistic free electron is ( $\partial_\mu = \partial/\partial x^\mu = (\partial/\partial t, \vec{\partial})$ )

$$(i\gamma^\mu \partial_\mu - a)\Psi(x) = 0$$

where the nature of the coefficient  $\gamma_\mu$  and the constant  $a$  are not specified yet.

Energy and momentum of the electron fulfill the relativistic energy-momentum

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relation

$$E^2 - \vec{p}^2 = m^2$$

where  $m$  is the mass of the electron. Thus, the components of the field function  $\Psi$  should obey the Klein-Gordon equation.

Let's linearize the Klein-Gordon equation as follows:

$$(-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\Psi(x) = 0$$

so that

$$(1/2(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)\partial_\mu \partial_\nu + m^2)\Psi(x) = 0$$

This equation is identical with the Klein-Gordon equation if

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

Dirac found that this relation can be fulfilled when assuming that  $\gamma_\mu$  are  $4 \times 4$  matrices. In the Dirac representation these Dirac matrices read (1 denotes the  $2 \times 2$  unit matrix):

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

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Finally, Dirac postulates the following equation for a relativistic electron

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0$$

where  $\Psi(x)$  is a field function with four components  $\Psi_i(x)$  which fulfill the Klein-Gordon equation.

$\Psi(x)$  is called a Dirac spinor, i.e. it transforms under transformations of the Lorentz (Poincare) group as follows

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$$

$$\Psi(x) \rightarrow \Psi'(x') = S(\Lambda)\Psi(x)$$

The Dirac matrices transform as four-vectors

$$\gamma'^\mu = \Lambda^\mu_\nu \gamma^\nu = S^{-1} \gamma^\mu S$$

One can show that there exists such a matrix  $S(\Lambda)$  so that  $\Psi'(x')$  also fulfills the Dirac equation

$$(i\gamma'^\mu \partial'_\mu - m)\Psi'(x') = 0$$

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## 3.3 From classical to quantum fields

Quantization schemes:

- Canonical quantization

It closely mimics the development of QM, i.e. time is singled out as a special coordinate. For complicated theories, such as non-Abelian theories, it is very tedious.

- Path integral method

An elegant and powerful quantization which is more suitable when quantizing non-Abelian theories. However, the mathematics of functional derivatives is quite involved and may not even exist in Minkowski space.

**Canonical quantization on the example of a free scalar real-valued Klein-Gordon field:**

Define the canonical conjugate fields (conjugate momenta) as

$$\Pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)} = \partial_t \Phi$$

**Transition to quantum fields:**

- 1. Re-interpret the fields  $\Phi$  and the conjugate momenta as hermitean operators

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in the Heisenberg picture (in the Hilbert space of state vectors).

- 2. Postulate the following commutation relations:

$$[\Phi(t, \vec{x}), \Pi(t, \vec{y})] = i\delta^3(\vec{x} - \vec{y})$$

and all commutators between  $\Phi$  itself and  $\Pi$  itself are zero.

How can we satisfy this relation and what are the direct consequences ?