4.3 How do the electroweak gauge bosons acquire mass ?

Explicit mass terms in the Lagrangian break gauge invariance.

A solution has been suggested by Weinberg and Salam (based on the work of Brout, Englert, Guralnik, Hagen, Higgs and Kibble) using the phenomenon of *spontaneous symmetry breaking or hidden symmetries*:

A symmetry of the basic equations describing a physical system is not a symmetry of the ground state of the system.

Spontaneous symmetry breaking is connected to the occurrence of degenerate ground states.

Let's consider a Lagrangian of a multiplet of self-interacting real-valued scalar fields $\Phi_i(x)$ with potential $V(\Phi)$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_i)(\partial^\mu \Phi_i) - V(\Phi_i)$$

that is invariant under transformations of Φ_i of a global continuous symmetry group G:

$$\Phi_i(x) \to \Phi_i'(x) = \Phi_i(x) + i\delta\theta^a T^a_{ij} \Phi_j(x)$$

where $T^a_{ij}, a = 1, \ldots N$ are the N generators of G. Thus,

$$0 = \delta V = \frac{\partial V}{\partial \Phi_i} \delta \Phi_i = i\epsilon^a \frac{\partial V}{\partial \Phi_i} T^a_{ij} \Phi_j = 0$$

After taking the derivative we find that the invariance of $\mathcal{L}$ under G requires

$$\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k} T^a_{ij} \Phi_j + \frac{\partial V}{\partial \Phi_i} T^a_{ik} = 0$$

Consider the case where the potential V has minima at $\Phi_i \equiv v_i \neq 0$, i.e. the ground state (corresponding to the vacuum expectation value of the quantized system) is not zero. Then we find

$$\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k} |_{\Phi_i = v_i} T^a_{ij} v_j = 0 \quad (*)$$

and for small oscillations around the vacuum state we can write

$$V(\Phi_i) = V(v_i) + \frac{1}{2} \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k} |_{\Phi_i = v_i} (\Phi_i - v_i)(\Phi_k - v_k) + \ldots$$

where it becomes transparent that

$$(M^2)_{ik} = \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k}|_{\Phi_i = v_i}$$

are mass terms of the (shifted) fields $\Phi'_i = \Phi_i - v_i$.

In general there may exist a subgroup H of G under which the ground state is invariant

$$\hat{T}^{\hat{a}}_{ij} v_j = 0 \quad \hat{a} = 1, ..., \hat{N}$$

Then $\hat{N}$ of the N equations of $(*)$ are already trivially fulfilled and $M^2 > 0$. The remaining $N - \hat{N}$ equations of $(*)$ require eigenvalues of $M^2 = 0$. The corresponding field excitations are called *Goldstone bosons*.

What happens if we consider a theory with local non-abelian gauge symmetry, where the scalar fields $\Phi_i(x)$ couple "minimally" to the gauge bosons ?

The gauge bosons of the broken symmetry become massive and no Goldstone bosons occur, i.e the Goldstone modes are transformed into the longitudinal components of the gauge fields.

Let's consider the Lagrangian of a self-interacting complex valued scalar field

(Higgs field) represented as a $SU(2)_L$ doublet ($\Phi_{1,2}$ are complex valued):

$$\Phi(x) = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix}$$

which is invariant under transformations of the local $SU(2)_L \times U(1)_Y$ symmetry group

$$\mathcal{L}_H = (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) - V(\Phi)$$

$$V(\Phi) = -\mu^2 (\Phi^\dagger \Phi) + \lambda (\Phi^\dagger \Phi)^2 \ , \ \lambda > 0$$

$$\mathcal{D}_\mu = \partial_\mu + ig\frac{\tau^a}{2} W_\mu^a + ig' \frac{Y_H}{2} B_\mu$$

Spontaneous symmetry breaking (SSB) occurs when we choose $\mu^2 > 0$. Desired pattern of SSB:

$$SU(2)_L \times U(1)_Y \to U(1)_{em}$$

so that the three gauge bosons of SU(2) become massive (SU(2) completely spontaneously broken) and the $U(1)_{em}$ gauge boson, the photon, remains massless (vacuum is still invariant under $U(1)_{em}$). Thus, we choose $\Phi_2 = \Phi^0, Q = 0, Y_H = 1$ and $\Phi_1 = \Phi^+$.

The potential V is minimized by a constant field $\Phi^\dagger\Phi = v^2/2 = \mu^2/2/\lambda$. At first approximation, the vacuum expectation value of the quantized system is given by the ground state of the classical potential

$$| < 0|\Phi|0 > | = \frac{v}{\sqrt{2}} \quad ; \; v = \frac{\mu}{\sqrt{\lambda}}$$

The orientation of the ground state in the weak isospin space is not fixed

$$< 0|\Phi|0 >= e^{iT^a\xi^a} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

The choice of one ground state breaks the SU(2) symmetry spontaneously (i.e. only the vacuum state is no longer invariant under SU(2) transformations)

$$T^a \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \neq 0$$

while it is still $U(1)_{em}$ invariant

$$Q \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = 0$$

Consider excitations around the vacuum state and replace Φ with $\Phi' = \Phi - v$ with per construction $< 0|\Phi'|0 >= 0$. Then we find that only the combination $\Phi_2^{'\dagger} + \Phi_2'$ becomes massive

$$V(\Phi') \propto \lambda \frac{v^2}{2} (\Phi_2^{'\dagger} + \Phi_2')^2$$

and the remaining three Higgs field excitation modes are massless and are called Goldstone bosons (Goldstone theorem: $G = SU(2)$ completely broken $\Rightarrow N = 3, \hat{N} = 0 \Rightarrow 3$ Goldstone bosons).

What happens to these Goldstone modes when we couple the Higgs field to the

SU(2) gauge fields in a gauge invariant way via minimal substitution (see above) ?

The Higgs field is described by a SU(2) invariant Lagrangian. So we can always parametrize the Higgs field as follows:

$$\Phi(x) = e^{iT^a \xi^a(x)} \left(\begin{array}{c} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{array} \right)$$

with $< 0|\xi^a(x)|0 >= 0$ and $< 0|\eta(x)|0 >= 0$. Choosing the unitary gauge, i.e. no would-be Goldstone bosons ξ^a occur:

$$\Phi(x) = \left(\begin{array}{c} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{array} \right)$$

and changing basis in fields by rotating to the physical gauge fields (mass eigenstates) we find mass terms for the electroweak gauge bosons, W^+, W^-, Z^0:

$$\mathcal{L}_H = M_W^2 W^{\mu -} W_\mu^+ + \frac{1}{2} M_Z^2 Z_\mu Z^\mu - \frac{1}{2} M_\eta^2 \eta^2 + \ldots$$

$$M_W = \frac{vg}{2} \quad ; \quad M_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$$

and no mass term for the photon field A_μ. There we had to choose $g \sin\theta_w = g' \cos\theta_w$. No Goldstone bosons occur. Only one physical massive scalar boson survives – the Higgs boson with $M_\eta = \frac{\sqrt{\lambda}}{2} v$. Moreover, we can obtain mass terms for the fermions in a gauge invariant way by their interaction with the Higgs field (Yukawa coupling).

Finally, here is the complete Lagrangian of the GWS model, which is part of the Standard Model Lagrangian describing leptons and their electroweak interaction (in unitary gauge, i.e. no would-be Goldstone bosons occur):

$$\mathcal{L}_{GWS} = \sum_f (\bar{\Psi}_f (i\gamma^\mu \partial\mu - m_f)\Psi_f - eQ_f \bar{\Psi}_f \gamma^\mu \Psi_f A_\mu) +$$

$$+\frac{g}{\sqrt{2}} \sum_i (\bar{a}_L^i \gamma^\mu b_L^i W_\mu^+ + \bar{b}_L^i \gamma^\mu a_L^i W_\mu^-) + \frac{g}{2c_w} \sum_f \bar{\Psi}_f \gamma^\mu (I_f^3 - 2s_w^2 Q_f - I_f^3 \gamma_5) \Psi_f Z_\mu +$$

$$-\frac{1}{4}|\partial_\mu A_\nu - \partial_\nu A_\mu - ie(W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 - \frac{1}{2}|\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ +$$

$$-ie(W_\mu^+ A_\nu - W_\nu^+ A_\mu) + ig' c_w (W_\mu^+ Z_\nu - W_\nu^+ Z_\mu|^2 +$$

$$-\frac{1}{4}|\partial_\mu Z_\nu - \partial_\nu Z_\mu + ig' c_w (W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)|^2 +$$

$$-\frac{1}{2} M_\eta^2 \eta^2 - \frac{g M_\eta^2}{8 M_W} \eta^3 - \frac{g'^2 M_\eta^2}{32 M_W} \eta^4 + |M_W W_\mu^+ + \frac{g}{2} \eta W_\mu^+|^2 +$$

$$+\frac{1}{2}|\partial_\mu \eta + i M_Z Z_\mu + \frac{ig}{2c_w} \eta Z_\mu|^2 - \sum_f \frac{g}{2} \frac{m_f}{M_W} \bar{\Psi}_f \Psi_f \eta$$

where $\Psi_f(x)$ is the Dirac spinor of fermion $f = e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$, $\Psi_L(x) = (a_L^i(x), b_L^i(x)), i = 1, 2, 3$ denotes the $SU(2)_L$ doublet of left-handed fermions with a_L^i with $I_i^3 = 1/2$ (neutrinos) and b_L^i with $I_i^3 = -1/2$ (charged leptons), I_i^3 denotes the 3.component of the weak isospin, Q_f the electric charge, $s_w = \sin\theta_w$, $c_w = \cos\theta_w$ with θ_w denoting the weak mixing angle, g is the $SU(2)_L$ gauge coupling constant, g' is the $U(1)_Y$ gauge coupling constant, $e = \sqrt{4\pi\alpha}$ with α being the fine structure constant, $\eta(x)$ denotes the Higgs field, $A_\mu(x)$ the electromagnetic field, $W_\mu^\pm(x), Z_\mu(x)$ are the 3 weak gauge fields. m_f, M_W, M_Z, M_η denote the fermion, W boson, Z boson and Higgs boson masses, respectively,

The fermion masses in $\mathcal{L}_{GSW}$ result from their Yukawa interaction with the Higgs field ($\tilde{\Phi} = i\sigma^2\Phi^* = (\Phi^{0*}, -\Phi^-)$)

$$\mathcal{L}_{Yukawa} = \sum_{ij} c_{ij} \bar{\Psi}_L^i b_R^j \Phi + \sum_{ij} \tilde{c}_{ij} \bar{\Psi}_L^i a_R^j \tilde{\Phi} + h.c.$$

with $b_R^i, b^i = e, \tau, \mu$ are SU(2) singlets of right-handed leptons. With $\Phi = (0, v + \eta)/\sqrt{2}$ and after diagonalization of the mass matrix the fermion mass terms arise with $m_f = v/\sqrt{2} c_i^H, c_{ij} = \delta_{ij} c_i^H$.