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### 3.4 The QED Lagrangian

After having quantized the free Dirac particle, as a next step towards a QFT of electromagnetic interaction among fermions, we need to couple the spin 1/2 Dirac particle,  $\Psi(x)$ , to a spin 1 Maxwell field  $A_\mu$ .

With the help of the electromagnetic field strength tensor  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ , where  $A^\mu$  denotes the electromagnetic four-potential  $A^\mu = (V, \vec{A})$  with

$$\vec{E} = -\vec{\nabla}V - \frac{\partial}{\partial t}\vec{A}; \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

Maxwell's equations can be expressed compactly in covariant form as follows:

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\epsilon_{\mu\nu\rho\sigma} \partial_\nu F^{\rho\sigma} = 0$$

with the four-current  $j^\mu = (\rho, \vec{j})$ . An appropriate Lagrangian for the free electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

from which Maxwell's equations in the absence of sources can be derived.

The electromagnetic potential is not uniquely defined:

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Any electromagnetic potential  $\tilde{A}^\mu = A^\mu + \partial^\mu \Lambda(x)$  describes the same electromagnetic field strength tensor. This freedom is called *gauge invariance*.

The idea of using gauge invariance as a dynamical principle originates from H. Weyl's attempts of finding a common geometric basis for both electromagnetism and gravity by requiring invariance under a space-time dependent change of scale (*Eichinvarianz*, Eich=gauge, standard of calibration).

The requirement of gauge invariance, i.e. invariance of the Dirac Lagrangian under local phase rotations

$$\begin{aligned}\Psi(x) &\rightarrow \Psi(x)e^{iq\alpha(x)} \\ \mathcal{L} \rightarrow \mathcal{L} &= \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - q\bar{\Psi}(x)\gamma^\mu \partial_\mu \alpha(x)\Psi\end{aligned}$$

necessitates the introduction of a vector field  $A_\mu$  (spin 1) which transforms as follows

$$A_\mu \rightarrow A_\mu - \partial_\mu \alpha(x)$$

and, following the *minimal coupling prescription*, couples to the Dirac field via the gauge covariant derivative

$$\partial_\mu \rightarrow \mathcal{D}_\mu = \partial_\mu + iqA_\mu.$$

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The vector field is called a gauge field and the associated particles are called gauge bosons.

After identifying  $A_\mu$  as the electromagnetic four-potential and  $q$  as the electric charge, the kinetic energy of the vector field, which describes the propagation of free photons, is added, and we find the **QED Lagrangian**

$$\mathcal{L} = \bar{\Psi}(x)(i\gamma^\mu \mathcal{D}_\mu - m)\Psi(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

which is invariant under local phase rotations. Note that a mass term for the photon  $m^2 A^\mu A_\mu$  would violate local gauge invariance.

We obtain the equations of motion from Hamilton's principle of least action: the Euler-Lagrange equation for  $\Psi(x)$  yields

$$(i\gamma^\mu \mathcal{D}_\mu - m)\Psi(x) = 0$$

and for  $A_\nu$

$$\partial_\mu F^{\mu\nu} = e\bar{\Psi}\gamma^\nu\Psi = j^\nu$$

which is the inhomogeneous Maxwell equation with the current density given by the conserved Noether vector current.