3. QED – First example of a QFT with direct experimental applications

Foundations: see e.g. S.Weinberg, Nobel lecture (1980)

- Local relativistic Quantum Field Theory (Group Theory + Quantum Mechanics)
- Symmetry principles
- Principle of renormalizability

An ansatz for the action (or the Lagrangian density) can be regarded as a formulation of the theory

$$S = \int_{\Omega} d^4x \mathcal{L}$$

To be a physically relevant theory the action (Langrangian) has to satisfy the general principles.

All fields transform as (irreduzible) representations of the Lorentz (Poincare) and some isospin groups.

The theory is unitary and the action is causal, renormalizable and an invariant under these groups.

From the action we obtain

- equations of motion (Hamilton's principle)
- conservation laws (Noether's theorem)
- transition from classical to quantum physics (using canonical or path integrals quantization)

3.1 Lagrange formalism and Symmetries

The construction of a field theory may begin from the *Lagrangian density*, $\mathcal{L}$, where $\mathcal{L}$ is a functional of the field $\Phi(x)$, and its four-gradient $\partial_\mu \Phi(x)$.

The classical action is defined as

$$S = \int_\Omega d^4x \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))$$

The equation of motions are obtained by Hamilton's principle of *least action*: The dynamics of the system evolving from the initial state $\Phi(t_1, \vec{x})$ to the final state $\Phi(t_2, \vec{x})$ is such that the action remains stationary during the evolution, $\delta S = 0$. This requirement is ensured by the Euler-Lagrange equations of motion for the classical fields

$$\frac{\partial \mathcal{L}}{\partial \Phi(x)} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi(x))}$$

Conservation laws observed in Nature may be imposed as symmetries of the Lagrangian, restricting or prescribing its form.

The equations of motion will be Lorentz-invariant, if the Lagrangian itself transforms as a Lorentz scalar.

Noether's theorem: conservation laws are correlated with continuous (infinitesimal) global symmetry transformations under which the Lagrangian remains (form) invariant

$$\Phi \to \Phi'(x) = \Phi(x) + \delta\Phi(x)$$

$$\delta\mathcal{L} = \mathcal{L}'(\Phi'(x), \partial_\mu \Phi'(x)) - \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) = 0$$

with $\mathcal{L}'(\Phi'(x), \partial_\mu \Phi'(x)) = \mathcal{L}(\Phi'(x), \partial_\mu \Phi'(x))$ and the action remains numerically unchanged. Together with the equation of motions this leads to conserved currents $\partial_\mu j^\mu = 0$ and a constant of motion provided the currents fall off sufficiently rapidly at the space boundaries of Ω.

Example: Klein-Gordon equation

A simple field theory is the one of a complex-valued charged field Φ that transforms as a Lorentz scalar

$$\mathcal{L} = |\partial_\mu \Phi|^2 - m^2 |\Phi|^2$$

leading to the Klein-Gordon equations for a spinless free particle of mass m

$$(\partial_\mu \partial^\mu + m^2)\Phi(x) = 0 \text{ and } (\partial_\mu \partial^\mu + m^2)\Phi^*(x) = 0$$

For a more sophisticated field theory this needs to be generalized by introducing spin and interactions.

A global transformation of these fields

$$\Phi \to e^{iq\alpha}\Phi \text{ and } \Phi^* \to e^{-iq\alpha}\Phi^*$$

leads to infinitesimal variations

$$\delta\Phi = iq\delta\alpha\Phi$$

Global phase invariance means that such transformations leave the Lagrangian unchanged (form invariant) and $\delta S = 0$

$$\delta S = \int d^4x \left\{ \frac{\delta \mathcal{L}}{\delta \Phi(x)} \delta\Phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi(x))} \delta(\partial_\mu \Phi) + \Phi \to \Phi^* \right\} = 0$$

Thus, we can identify a conserved Noether current

$$j^\mu = iq[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi(x))}\Phi - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^*(x))}\Phi^*]$$

which satisfies $\partial_\mu j^\mu = 0$. When identifying q as the electric charge, j^μ is the electromagnetic current of the charged scalar field.

From current conservation follows that the charge $Q = \int d^3x j^0$ is conserved ('constant of motion'):

$$0 = \int d^3x \partial_\mu j^\mu = \int d^3x \partial_0 j^0 - \int d^3x \partial_i j^i = \frac{d}{dt}\int d^3x j^0 - \text{surface term} = \frac{d}{dt}Q(t)$$