Bhabha Scattering at NNLO

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1. Why NNLO QED Corrections to Bhabha Scattering?

2. NNLO Electron Loop Corrections

3. NNLO Photonic Corrections

4. NNLO Heavy Flavor and Hadronic Loop Corrections

5. Conclusions
Bhabha Scattering and Luminosity-I

$e^+ e^- \rightarrow e^+ e^-$

\[ s \equiv -p^2 = -(p_1 + p_2)^2 = 4E^2 > 4m^2 \quad t \equiv -Q^2 = -(p_1 - p_3)^2 = -4(E^2 - m^2) \sin^2 \frac{\theta}{2} < 0 \]

- Effective tool for the Luminosity measurement @ $e^+ e^-$ colliders

\[ \sigma_{\text{exp}} \equiv \frac{N}{L} \quad L = \frac{N}{\sigma_{\text{bh-th}}} \]
In the region employed for $L$ measurements the Bhabha scattering cross section is large, QED dominated, and measured with very high precision (1 permille at KLOE/DAFNE).

SABH is employed at LEP and ILC, while LABH is employed at colliders operating at $\sqrt{s} = 1 - 10\text{GeV}$.

Due to beam-beam interactions, at ILC the colliding energy $\sqrt{s}$ shows a continuous spectrum: the LABH can also be used to determine the luminosity spectrum.

Realistic simulation of Bhabha events are performed by sophisticated MC generators which take into account the detector geometry, experimental cuts, theoretical input (fixed order calculations, resummation).
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The accuracy of the theoretical evaluation of the Bhabha scattering cross section directly affects the luminosity determination.

\[ \implies \text{study of radiative corrections to Bhabha scattering} \]
In this talk we consider the QED process only

- We consider differential cross-sections summed over the spins of the final state particles and averaged over the spin of the initial ones

\[
\frac{d\sigma_0(s, t)}{d\Omega} = \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[ st + \frac{s^2}{2} + (t - 2m^2)^2 \right] \right. + \frac{1}{t^2} \left[ st + \frac{t^2}{2} + (s - 2m^2)^2 \right] \\
\left. + \frac{1}{st} \left[ (s + t)^2 - 4m^4 \right] \right\}
\]
Virtual Corrections to the Cross Section - I

\[ \frac{d\sigma(s, t)}{d\Omega} = \frac{d\sigma_0(s, t)}{d\Omega} + \left( \frac{\alpha}{\pi} \right) \frac{d\sigma_1(s, t)}{d\Omega} + \left( \frac{\alpha}{\pi} \right)^2 \frac{d\sigma_2(s, t)}{d\Omega} + \mathcal{O}\left(\frac{\alpha}{\pi}^3\right) \]

The \(\mathcal{O}(\alpha^3)\) virtual corrections (one-loop × tree-level) are well known (in the full SM), no problem in keeping \(m_e \neq 0\)

M. Consoli (1979),
M. Böhm, A. Denner, and W. Hollik (1988),
M. Greco (1988),...

\[ \frac{\alpha}{\pi} \frac{d\sigma_1^V(s, t)}{d\Omega} = \frac{s}{16} \sum_{\text{spin}} \left\{ \left( \begin{array}{c} \text{diagram} \\ \text{with lines} \end{array} \right) \right\}^* \times \left( \begin{array}{c} \text{diagram} \\ \text{with lines} \end{array} \right) + \text{c.c.} + \cdots \]
Virtual Corrections to the Cross Section - II

Order $\alpha^4 QED$ corrections:

- Contributions from two-loop $\times$ tree-level and one-loop $\times$ one-loop
- Can be divided in three sets,
  i) with a closed electron loop,
  ii) closed heavy(er) flavor loop, and
  iii) photonic (without fermion loops)

\[
\left( \frac{1}{\pi} \right)^2 \frac{d\sigma}{d\Omega} = \frac{s}{16} \sum_{\text{spin}} \left\{ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array} - \begin{array}{c}
\text{Diagram 2}
\end{array} \right\}^* \times \begin{array}{c}
\text{Diagram 3}
\end{array} + \text{c.c.}
\]

\[
+ \left\{ \begin{array}{c}
\text{Diagram 4}
\end{array} - \begin{array}{c}
\text{Diagram 5}
\end{array} \right\}^* \times \begin{array}{c}
\text{Diagram 6}
\end{array} + \text{c.c.} + \cdots \right\}
\]
The virtual corrections were first obtained in the massless electron approximation

Z. Bern, L Dixon, and A. Ghinculov ('00)
**Radiative corrections in $m_e = 0$ approximation**

The virtual corrections were first obtained in the massless electron approximation.

Z. Bern, L Dixon, and A. Ghinculov ('00)

However, in order to interface the fixed order calculation with the existing MC, it is necessary to keep the electron mass as a collinear regulator.
Electron Loop Corrections
All the two-loop graphs including a closed electron loop can be calculated also keeping $m_e \neq 0$ and without relying on any approximation or expansion.
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The relevant integrals can be reduced to a combination of a relatively small set of Master Integrals employing the Laporta algorithm.

The MIs (including the ones for the box) can be evaluated employing the differential equation method.

R. Bonciani et al. (’03–’04)
What to do With the Integrals?

For each two-loop graph, after interfering it with the Born amplitude we are left with a linear combinations of integrals of the form

\[
\int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \frac{S_{n_1} \cdots S_{n_q}}{\mathcal{D}_{m_1} \cdots \mathcal{D}_{m_t}}
\]

\[
\begin{array}{ll}
k_i & \rightarrow \text{integration momenta} \\
p_i & \rightarrow \text{external momenta} \\
S & \rightarrow \text{scalar products } k_i \cdot k_j \text{ or } k_i \cdot p_k \\
\mathcal{D} & \rightarrow \text{propagators} \\
\end{array}
\]

\[
[\sum c_i k_i + \sum d_j p_j]^2 (+m_e^2)
\]

Luckily, just a “small” number of these diagrams are independent: the MIs

It is necessary to

- identify the MIs \(\Rightarrow\) Reduction through the Laporta Algorithm
- calculate the MIs \(\Rightarrow\) Differential Equation Method
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Multi-loop Technology: The Laporta Algorithm

The set of denominators $\mathcal{D}_1, \cdots, \mathcal{D}_t$ defines a topology; for each topology

- The scalar integrals are related via Integration By Parts identities (10 identities per integral for a two-loop four-point function)

$$\int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \frac{\partial}{\partial k_i^{\mu}} \left[ \nu^\mu S_1^{m_1} \cdots S_q^{m_q} \right] = 0 \quad \nu^\mu = k_1, k_2, p_1, \cdots, p_3$$

- Building the IBPs for growing powers of the propagators and scalar products the number of equations grows faster than the number of unknowns: one finds a system of equations which is apparently over-constrained

- Solving the system of IBPs (in a problem with a small number of scales) one finds that only a few of the scalar integrals above (if any) are independent: the MIs.
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\[
\int D^d k_1 D^d k_2 \frac{\partial}{\partial k_i^\mu} \left[ v^\mu S^{n_1}_1 \cdots S^{n_q}_q \frac{D^{m_1}_1}{D_t} \ldots \frac{D^{m_t}_t}{D_t} \right] = 0 \quad v^\mu = k_1, k_2, p_1, \ldots, p_3
\]

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- Building the IBPs for growing powers of the propagators and scalar products the number of equations grows faster that the number of unknown: one finds a system of equations which is apparently over-constrained

- Solving the system of IBPs (in a problem with a small number of scales) one finds that only a few of the scalar integrals above (if any) are independent: the MIs.
The two-loop box diagrams entering in the calculation of the electron-loop corrections are reducible, i.e. they can be rewritten in terms of integrals belonging to the subtopologies only:

\[ \text{Born Amplitude} \Rightarrow \sum C + \text{Triangles} + \text{Bubbles} + \text{Tadpoles} \]
Calculation of the MIs: Differential Equation Method

For each Master Integral belonging to a given topology
\[ F^{(q)}_l \rightarrow \{D_1, \cdots, D_q \} \]

- Take the derivative of a given integrals with respect to the external momenta \( p_i \)

\[ p_j^\mu \frac{\partial}{\partial p_i^\mu} F^{(q)}_l = p_j^\mu \int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \frac{\partial}{\partial p_i^\mu} \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_q^{m_q}} \]

- The integral are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs

- Rewrite the diff. eq. in terms of derivatives with respect to \( s \) and \( t \)

- Fix somehow the initial condition(s) (ex. knowing the behavior of the integral at \( s = 0 \)) and solve the system of DE(s)
Calculation of the MIs: Differential Equation Method

For each Master Integral belonging to a given topology $F_i^{(q)} \rightarrow \{D_1, \cdots, D_q\}$

- Take the derivative of a given integral with respect to the external momenta $p_i$
- The integral are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs

$$p_j^\mu \int D^d k_1 D^d k_2 \frac{\partial}{\partial p_i^\mu} \frac{S_{1}^{n_1} \cdots S_{q}^{n_q}}{D_{1}^{m_1} \cdots D_{q}^{m_q}} = \sum c_i F_i^{(q)} + \sum_{r\neq q} \sum_j k_j F_j^{(r)}$$

- Rewrite the diff. eq. in terms of derivatives with respect to $s$ and $t$
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**Calculation of the MIs:**

**Differential Equation Method**

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- Rewrite the diff. eq. in terms of derivatives with respect to \( s \) and \( t \)

\[
\frac{\partial}{\partial s} F_i^{(q)}(s, t) = \sum_j c_j(s) F_j^{(q)}(s) + \sum_{r \neq q} \sum_l k_l(s) F_l^{(r)}(s)
\]

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Calculation of the MIs: 
**Differential Equation Method**

For each Master Integral belonging to a given topology

\[ F^{(q)}_I \rightarrow \{D_1, \cdots, D_q\} \]

- Take the derivative of a given integrals with respect to the external momenta \( p_i \)
- The integral are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs
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Five Denominator MIs

The most complicated irreducible topology in the calculation of the electron loop corrections is a five denominator box with two MIs (thick lines indicate massive propagators, thin lines massless ones)

\[ M(\alpha_1, \ldots, \alpha_9) = \int \frac{\mathcal{D}^d k_1 \mathcal{D}^d k_2 (p_1 \cdot k_1)^{\alpha_6} (p_1 \cdot k_2)^{\alpha_7} (p_2 \cdot k_2)^{\alpha_8} (p_4 \cdot k_2)^{\alpha_9} \cdot P_0^{\alpha_1}(k_1) P_m^{\alpha_2}(p_2 - k_1) P_m^{\alpha_3}(p_4 - k_1) P_m^{\alpha_4}(p_1 + p_2 - k_1 + k_2) P_m^{\alpha_5}(k_2)}{P_m^\alpha(p_2 - k_1) P_m^\alpha(p_4 - k_1) P_m^\alpha(p_1 + p_2 - k_1 + k_2)} \]

\[ P_0(q) = q^2 \quad \quad P_m(q) = q^2 + m^2 \]
Five Denominator MIs-II

\[ M_1 = \quad (p_1 \cdot k_2) \quad M_2 = \]
The two MIs satisfy two independent first order differential equations:

\[
\frac{dM_i(s, t)}{dt} = C_i(s, t)M_i(s, t) + \Omega_i(s, t)
\]
\[ M_1 = \quad M_2 = (p_1 \cdot k_2) \]

- the two MIs satisfy two independent first order differential equations

\[
\frac{dM_i(s, t)}{dt} = C_i(s, t)M_i(s, t) + \Omega_i(s, t)
\]

- the initial conditions are determined by setting \( p_2 = p_4 \) (special case of \( t = 0 \)): one obtains a three-point function
Analytic Expression for the MIs

By solving the differential equation(s) it is possible to obtain analytic expressions for the MIs \((s = -(1 - x)^2/x, \ t = -(1 - y)^2/y)\)

\[
M_1(d, m, P, Q) = \frac{M_1^{(-2)}}{(d - 4)^2} + \frac{M_1^{(-1)}}{(d - 4)} + M_1^{(0)}
\]

\[
m^2 M_1^{(-2)} = \frac{1}{8} \left[ \frac{1}{1 - x} - \frac{1}{1 + x} \right] H(0; x)
\]

\[
m^2 M_1^{(-1)} = \frac{1}{16} \left[ \frac{1}{1 - x} - \frac{1}{1 + x} \right] \{ \zeta(2) - [2 - (1 - \frac{2}{1 - y})] H(0; y) \} H(0; x)
\]

\[
m^2 M_1^{(0)} = -\frac{1}{16} \left[ \frac{1}{1 - x} - \frac{1}{1 + x} \right] \{ \zeta(2) + \zeta(3) - 2H(0; x) - \zeta(2)H(-1; x)
\]

\[
- H(0, 0; x) + 2H(-1, 0; x) - H(0, 0, 0; x) + H(-1, 0, 0; x)
\]

\[
+ H(0, 0; y) H(0; x) - 2H(-1, -1, 0; x) + H(0, -1, 0; x)
\]

\[
+ H(0, 0; y) H(0; x) - \frac{1}{2} \left[ 1 - \frac{2}{1 - y} \right] \left[ 4\zeta(2) H(0; y) + \cdots \right]
\]
Functions of the variable $x$ and a set of indices $\vec{a}$ with weight $w$; each index can assume values $1, 0, -1$

$$H(a; x)$$

Definitions: $w = 1$

$$H(1; x) = \int_0^x \frac{dt}{1 - t} = -\ln(1 - x)$$

$$H(0; x) = \ln x$$

$$H(-1; x) = \int_0^x \frac{dt}{1 + t} = \ln(1 + x)$$

$$\frac{d}{dx}H(a; x) = f(a; x) \quad f(1; x) = \frac{1}{1 - x} \quad f(0; x) = \frac{1}{x} \quad f(-1; x) = \frac{1}{1 + x}$$
**HPLs: Definitions**

Definitions: \( w > 1 \)

if \( \bar{a} = 0, 0, \ldots, 0 \) (\( w \) times) \[
H(\bar{0}_w; x) = \frac{1}{w!} \ln^w x
\]

else \[
H(i, \bar{a}; x) = \int_0^x dt f(i; t) H(\bar{a}; t)
\]

consequences: \[
\frac{d}{dx} H(i, \bar{a}; x) = f(i; x) H(\bar{a}; x) \quad H(\bar{a} \notin \bar{0}; 0) = 0
\]

a few examples @ \( w = 2 \)

\[
H(0, 1; x) = \int_0^x dt f(0; t) H(1; t) = -\int_0^x dt \frac{1}{t} \ln (1 - t) = \text{Li}_2(x)
\]

\[
H(1, 0; x) = \int_0^x dt f(1; t) H(0; t) = \int_0^x dt \frac{1}{1 - t} \ln t
\]

\[
= -\ln x \ln (1 - x) + \text{Li}_2(x)
\]
HPLs as a Generalization of the Nielsen’s PolyLogs

The HPLs include the Nielsen’s PolyLogs

\[ S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n+p)!p!} \int_0^1 \frac{dt}{t} \ln^{n-1} t \ln^p (1 - xt) \]

\[ \text{Li}_n(x) = S_{n-1,1}(x) \]

\[ \text{Li}_n(x) = H(\vec{0}_{n-1}, 1; x) \]

\[ S_{n,p}(x) = H(\vec{0}_n, \vec{1}_p; x) \]

but the HPLs are a larger set of functions: from \( w = 4 \) one finds things as

\[ H(-1, 0, 0, 1; x) = \int_0^x \frac{dt}{1 + t} \text{Li}_3(x) \notin \sum \text{Nielsen’s PolyLogs} \]
The HPLs Algebra

- **Shuffle Algebra:**
  \[ H(\tilde{p}; x)H(\tilde{q}; x) = \sum_{\bar{r}=\tilde{p} \oplus \tilde{q}} H(\bar{r}; x) \]

  some examples

  \[ H(a; x)H(b; x) = H(a, b; x) + H(b, a; x) \]
  \[ H(a; x)H(b, c; x) = H(a, b, c; x) + H(b, a, c; x) + H(b, c, a; x) \]

- **Product Ids:**
  \[ H(m_1, \ldots, m_q; x) = H(m_1; x)H(m_2, \ldots, m_q; x) \]
  \[ - H(m_2, m_1; x)H(m_3, \ldots, m_q; x) \]
  \[ + \cdots + (-1)^{q+1} H(m_q, \ldots, m_1; x) \]
2-dimensional Harmonic Polylogarithms (2dHPLs)


As for the HPLs, they are obtained by repeated integration over a new set of factors depending on a second variable.

\[ f(-y; x) = \frac{1}{x + y} \quad f(-1/y; x) = \frac{1}{x + 1/y} \]

\[ G(i, \tilde{a}; x) = \int_0^x dt f(i; t) G(\tilde{a}; t) \]

A few examples:

\[ G(-y; x) = \int_0^x \frac{dz}{z + y} = \ln \left(1 + \frac{x}{y}\right) \quad G(-1/y; x) = \int_0^x \frac{dz}{z + 1/y} = \ln (1 + xy) \]

\[ G(-y, 0; x) = \ln x \ln \left(1 + \frac{x}{y}\right) + \text{Li}_2 \left(-\frac{x}{y}\right) \]
The 2dHPLs share the properties of the HPLs. Up to \( w = 3 \) (our case) the 2dHPLs can be expressed in terms of \( \ln, \text{Li}_2, \text{Li}_3, S_{1,2} \).

The analytic properties of both HPLs & 2dHPLs are known. Codes for their numerical evaluation are available.

All of the steps in the calculation must be automated as much as possible.
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A (partial & personal) list of tools

- Diagram generation → QGRAF
- Squared matrix elements, Dirac (and color) algebra → FORM
- Reduction → Solve, Repete (C + MAPLE + FORM); AIR (MAPLE)
- Differential Equation Method → FORM and/or MATHEMATICA
- Initial conditions, tests of the MIs → AMBRE + MB (MATHEMATICA); FIESTA (MATHEMATICA); HPL and HypExp (MATHEMATICA)
- Final analytical results → FORM
- Numerical evaluation of the results → FORTRAN or MATHEMATICA; HPL (MATHEMATICA now also in C++)
In the electron loop corrections to the CS

- both UV and IR divergences are regularized within the DIM REG scheme
- the UV renormalization is carried out in the on-shell scheme
- the graphs are at first calculated in the non physical region $s < 0$ and then analytically continued to the physical region $s > 4m_e^2$
- the cross section can be expressed in terms of HPLs and 2dHPLs with arguments

\[
x = \frac{\sqrt{s} - \sqrt{s - 4m_e^2}}{\sqrt{s} + \sqrt{s - 4m_e^2}} \quad y = \frac{\sqrt{4m_e^2 - t} - \sqrt{-t}}{\sqrt{-t} + \sqrt{4m_e^2 - t}} \quad z = \frac{\sqrt{4m_e^2 - u} - \sqrt{-u}}{\sqrt{-u} + \sqrt{4m_e^2 - u}}
\]

- The residual IR poles are eliminated by adding the contribution of the soft photon radiation
After UV renormalization, the virtual CS still includes poles in $D - 4$, of IR origin, that can be eliminated by adding the contribution of the soft photon emission diagrams

In order to cancel the IR divergent terms in the virtual cross section at NLO and in the NNLO electron-loop corrections it is sufficient to consider the contribution of the single photon emission graphs

$$e^- (p_1) + e^+ (p_2) \longrightarrow e^- (p_3) + e^+ (p_4) + \gamma (k) \quad k_0 < \omega$$
The soft photon emission cross section is given by the interference of the tree level soft emission with

\[ \frac{d\sigma_2^S(s, t, m^2)}{d\Omega} = \frac{d\sigma_1^D(s, t, m^2)}{d\Omega} \sum_{i,j=1}^{4} J_{ij} \]

Soft emission corrections factor in the product of the one-loop cross section and a radiator factor \( J \)

Remember: \( \sigma_1^D \) is finite (after UV renormalization)
It is possible to understand how the cancellation of the IR poles works from a diagrammatic point of view:

\[ J_{12} + (J_{13} + J_{11}) = \text{IR fin} \]
Photonic Corrections
\( \mathcal{O}(\alpha^4) \) Photonic Corrections

With the same techniques employed in obtaining the \( \mathcal{O}(\alpha^4(N_F = 1)) \) non-approximated differential CS, it is possible to calculate the photonic virtual corrections (and related soft photon emission) to the CS at order \( \mathcal{O}(\alpha^4) \), except for the ones arising from the two loop photonic boxes.

The one- and two-loop Dirac form factors in the \( t \)-channel are sufficient to determine completely the small angle cross section

\[
\frac{d\sigma_2}{d\sigma_0} = 6(F_1^{(1l)}(t))^2 + 4F_1^{(2l)}(t)
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$\mathcal{O}(\alpha^4)$ Photonic Corrections

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$$\frac{d\sigma_2}{d\sigma_0} = 6(F_1^{(1l)}(t))^2 + 4F_1^{(2l)}(t)$$
(IR)Relevance on the Terms $\propto m_e^2$

\[ D_{\text{Vert}} = \left( \frac{\alpha}{\pi} \right)^2 \left| \frac{d\sigma_2^{(\text{Vert})}}{d\Omega} - \frac{d\sigma_2^{(\text{Vert})}}{d\Omega} \right|_L \]

\[ D_{\text{BoxBox}} = \left( \frac{\alpha}{\pi} \right)^2 \left| \frac{d\sigma_2^{(\text{Vert})}}{d\Omega} - \frac{d\sigma_2^{(\text{Vert})}}{d\Omega} \right|_L \]
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the terms proportional to $m_e$ become negligible for values of $E$ that are very small with respect to the ones encountered in $e^+e^-$ experiments
Building on the BDG result and on works by A. B. Arbuzov et al., B. Tausk, N. Glover, and J. J. van der Bij (’01) obtained the terms proportional to $L = \ln m_e^2/s$ of the full (virtual + soft) photonic CS (i.e. graphs including a closed electron loop have been neglected).

A. Penin (’05) obtained also the constant terms of the photonic CS in the $m_e^2/s$ expansion.

Therefore, in the expansion

$$
\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)} \ln^2 \left( \frac{s}{m_e^2} \right) + \delta_2^{(1)} \ln \left( \frac{s}{m_e^2} \right) + \delta_2^{(0)} + O \left( \frac{m_e^2}{s} \right)
$$

$\delta_2^{(2)}$, $\delta_2^{(1)}$, and $\delta_2^{(0)}$ are known.

Several partial cross-checks of this results were possible by comparing it with the $m_e^2/s \rightarrow 0$ limit of the exact result for the photonic vertex and one-loop by one-loop corrections.
Penin’s technique (in a Nutshell)

» Consider the amplitude of the two loop virtual corrections to the cross-section in which collinear and soft divergencies are regularized by $m_e$ and $\lambda$: $A^{(2)}(m_e, \lambda)$

» Build an auxiliary amplitude $\overline{A}^{(2)}(m_e, \lambda)$ with the same IR singularities of the $A^{(2)}(m_e, \lambda)$ but sufficiently simple to be evaluated in the small mass expansion

» The quantity $\delta A^{(2)} = A^{(2)} - \overline{A}^{(2)}$ has a finite limit when $m_e$ and $\lambda$ tend to zero

» $\delta A^{(2)}$ is regularization scheme independent and it can be reconstructed from the known results for the virtual corrections calculated by setting $m_e = \lambda = 0$ from the start
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A^{(2)} = \overline{A}^{(2)}(m_e, \lambda) + \delta A^{(2)} + O(m_e, \lambda)
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The method cannot be applied to the electron loop corrections
As can be seen from Penin’s result, when neglecting positive powers of the electron mass, the problem is equivalent to change the regularization scheme for the collinear singularities:

Is it possible to calculate graphs employing DIM REG to regulate both soft and collinear singularities and then translate a posteriori the collinear poles into collinear logs?
Mass from Massless-I

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Is it possible to calculate graphs employing DIM REG to regulate both soft and collinear singularities and then translate a posteriori the collinear poles into collinear logs?

For a generic QED/QCD process, with no closed fermion loops

\[ M^{(m \neq 0)} = \prod_{i \in \{\text{all legs}\}} Z_i^{\frac{1}{2}}(m, \varepsilon) M^{(m = 0)} \]

where \( Z \) is defined through the Dirac form factor

\[ F^{(m \neq 0)}(Q^2) = Z(m, \varepsilon) F^{(m = 0)}(Q^2) + O(m^2/Q^2) \]

A. Mitov and S. Moch ('06)
with a similar technique applied to Bhabha scattering it was possible to calculate all the NNLO corrections in the limit $s, |t|, |u| \gg m_f^2 \gg m_e^2$

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s} \left( \frac{1 - r + r^2}{r} \right) \left[ 1 + \frac{\alpha}{\pi} \delta_1 + \left( \frac{\alpha}{\pi} \right)^2 \delta_2 \right] \]

\( r = 1/2(1 - \cos \theta) \)

\[ \delta_2 = \delta_{2_{\text{photonic}}} + \delta_{2_{\text{electron loop}}} + \delta_{2_{\text{heavy flavor loop}}} \]

- photonic corrections in agreement with A. Penin ('05)
- electron loop corrections in agreement with R. Bonciani et al ('04)
- “heavy flavor” loop corrections in agreement with S. Actis et al ('07)
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T. Becher and K. Melnikov ('07)

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Heavy Fermion Corrections
Beyond $s \gg m_f^2$

In any realistic case the approximation $s, |t|, |u| \gg m_e^2$ is more than enough. However, in the case of corrections with a closed heavy fermion loop, it is not always true that $s, |t|, |u| \gg m_f^2$. 
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for example

- \( \tau \) loop at KLOE, where \( \sqrt{s} = 1 \text{GeV} < m_\tau \)
- top quark loop at ILC, where \( \sqrt{s} \approx 500 \text{GeV} \) and \( m_t^2/t, m_t^2/u < 1 \) just in the angular region \( 40^\circ < \theta < 140^\circ \)
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It is necessary to calculate the NNLO corrections including an heavy fermion loop by retaining the exact dependence on $m_f$

$$s, |t|, |u|, m_f^2 \gg m_e^2$$

this is a non trivial problem involving four-scale two-loop boxes ...
What is the collinear structure of these corrections?

\[ \delta_2 = \delta_2^C(s, t, m_{\bar{f}}^2) \ln \left( \frac{s}{m_{\bar{e}}^2} \right) + \delta_2^R(s, t, m_{\bar{f}}^2) + \mathcal{O}\left( \frac{m_{\bar{e}}^2}{s} \right) \]

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there is just a single collinear logarithm

It is possible to show that the collinear logarithm arises from trivial reducible graphs only
In a physical (Coulomb or axial) gauge, the collinear divergencies factorize and can be reabsorbed in the external field renormalization

J. Frenkel and J. C. Taylor (’76)
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In Bhabha scattering, the box (and the photon self-energy) diagrams form gauge-independent sets.
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Therefore, the boxes must form a collinear safe set in any gauge.
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Therefore the boxes must form a collinear safe set in any gauge.

In the Feynman gauge we employ in the calculation, individual box diagrams show collinear divergencies that cancel in the sum over all the box diagrams.
The sum of the one-particle irreducible diagrams has a regular behavior in the small electron mass $m_e$. 
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This means that we can set $m_e = 0$ from the beginning, getting rid of one scale
The Calculation of the Boxes

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\[ u = -s - t \]

= Free of collinear poles
The Calculation of the Boxes

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In the Feynman gauge

\[
\begin{align*}
\text{Diagram 1} & \quad + \quad \text{Diagram 2} \\
& = \text{Free of collinear poles}
\end{align*}
\]

\[u = -s - t\]

After UV renormalization, the only remaining poles are the IR (soft) ones.
It was possible to calculate the boxes for $m_e = 0$ and generic $s, |t|, |u|, m_f^2 \gg m_e^2$, therefore effectively eliminating one mass scale from the most challenging part of the calculation. We employed IBPs and Differential Eq. Method. The analytical result can be expressed in terms of HPL and a few GHPLs of a new class. The latter can be expressed in closed form in terms of polylogs. By expanding the exact result it was possible to recover the result of Actis et al and Becher Melnikov. It now is possible to study the $\tau$ loop effects at intermediate energies and top loop effects at ILC energies, where the $s, |t|, |u| \gg m_f^2$ approximation is not valid.
Another Five Denominator MIs

The most complicated irreducible topology in the calculation of the heavy-flavor loop corrections is a five denominator box with two MIs (thick lines indicate massive propagators, thin lines massless ones)

Like for the electron-loop MIs, the differential equation for the two MIs decouple

The initial conditions can be determined by imposing the regularity of the integral in $s = 0$
And now, for Something Completely Different...

Consider the diagram with a (massive or massless) quark loop contributing the top-quark pair production in the quark-antiquark channel: $q\bar{q} \rightarrow t\bar{t}$

R. Bonciani et al.
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R. Bonciani et al.

Two irreducible five-propagator box topologies:

1 MI

2 MIs
Heavy Flavor: Results

Two-loop corrections to the Bhabha scattering differential cross section at $\theta = 60^\circ$ due to a closed muon loop.

The large logs depending on the IR cut-off $\omega$ are excluded from the numerical analysis.

Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV due to a closed $\tau$-lepton, $c$-quark and $b$-quark loop for $m_c = 1.4$ GeV and $m_b = 4.7$ GeV.
Dispersion Relation Approach

Alternative way to calculate diagrams with vacuum polarization insertions

- replace the photon propagator on which one wants to insert a fermion loop as follows

\[
\frac{\delta_{\mu\nu}}{q^2} \longrightarrow \frac{\delta_{\mu\nu}}{q^2} (q^2 \delta_{\mu\nu} - q_\mu q_\nu) \prod (q^2) \frac{\delta_{\mu\nu}}{q^2}
\]

- apply the subtracted dispersion relation

\[
\Pi (q^2) = -\frac{q^2}{\pi} \int_{4m^2}^{\infty} dz \frac{\text{Im}\Pi(z)}{z} \frac{1}{q^2 + z}
\]

- relate the self energy imaginary part to the decay rate of an off-shell photon

\[
\text{Im}\Pi(z) = -\frac{3}{\alpha} R(z) \equiv -\frac{\alpha}{3} \sigma \left( e^+ e^- \rightarrow \gamma^* \rightarrow f\bar{f} \right)
\]

- perform first the integration over the photon momentum \(q\) and then the integration over \(z\)
The procedure based on dispersion relations can be applied to Bhabha scattering in a two-step calculation:

- analytic evaluation of the one-loop kernels with a “massive” photon
- numerical integration over \( z \)
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- numerical integration over $z$

**Hadronic Corrections**

The corrections related to the hadronic vacuum polarization can be obtained by taking $R_{\text{had}}(z)$ from $e^+e^- \rightarrow \text{hadrons}$ data.

At the ILC, hadronic corrections give the largest effect at large angles ($< 2\%$). At $\sqrt{s} \sim 1$ GeV they are $\sim 0.5$ per mille, At $\sqrt{s} \sim 10$ GeV they are $\sim 3$ per mille.
The actual impact of the two-loop virtual corrections on the theoretical predictions can be determined only after the corrections are implemented into MC event generators.

This was done only for photonic and electron loop corrections at \( \sqrt{s} \sim 1 - 10 \, \text{GeV} \) (BABAYAGA generator, G. Balossini et al. ('06))

- Non leading-log enhanced photonic corrections are of \( \sim 0.01 - 0.03\% \)
- Non leading-log enhanced electron-loop corrections are of \( \sim 0.01\% \)
- BABAYAGA claims a theoretical accuracy of the order of \( \sim 0.1\% \)
- the muon-loop corrections reach \( 0.05 \% \) of the tree level at \( \theta \sim 140 \)
- the tau, b, and c loop corrections are one order of magnitude smaller than the muon corrections
- The contribution of the light quarks u, d, s must be treated with the dispersion relation approach

S. Actis et al. ('07)
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S. Actis et al. ('07)
J. Kühn and S. Uccirati ('08)
A precise knowledge of the Bhabha scattering cross section (both at small and large angle) is crucial in order to determine the luminosity at $e^+e^-$ colliders.

Photonic and electron-loop corrections have been known for some time. Recently, we calculated also the heavy flavor NNLO corrections. The hadronic corrections were evaluated by two groups with an approach based on dispersion relations.

The NNLO QED corrections to Bhabha scattering were an ideal testing ground for calculational techniques which can be applied to other processes.

The calculation of the virtual NNLO QED corrections is basically complete, but there is still work to be done: ex. one-loop hard photon emission.

These fixed order results must be included/interfaced with MC generators.
Backup Slides
A scalar integral is invariant under Lorentz transformation of the external momenta:

\[ p^\mu \rightarrow p^\mu + \delta p^\mu = p^\mu + \delta \epsilon^\mu_\nu p^\nu \quad \delta \epsilon^\mu_\nu = -\delta \epsilon^\nu_\mu \]

\[ I(p_1, p_2, p_3) = I(p_1 + \delta p_1, p_2 + \delta p_2, p_3 + \delta p_3) \]

implying the following 3 identities for a 4-point functions:

\[ (p^\mu_1 p^\nu_2 - p^\nu_1 p^\mu_2) \sum_{n=1}^{3} \left[ p^\nu_n \frac{\partial}{\partial p^\mu_n} - p^\mu_n \frac{\partial}{\partial p^\nu_n} \right] I(p_i) = 0 \]

\[ (p^\mu_2 p^\nu_3 - p^\nu_2 p^\mu_3) \sum_{n=1}^{3} \left[ p^\nu_n \frac{\partial}{\partial p^\mu_n} - p^\mu_n \frac{\partial}{\partial p^\nu_n} \right] I(p_i) = 0 \]

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Further identities arise when a Feynman graph has symmetries.

It is in those cases possible to perform a transformation of the loop momenta that does not change the value of the integral but allows to express the integrand as a combination of different integrands.

Some relations are immediately seen:

\[ 0 = \begin{array}{c}
\text{graph 1} \\
- \text{graph 2}
\end{array} \]

Others are more involved:

\[ 0 = \begin{array}{c}
\text{graph 3} \\
+ \text{graph 4} \\
+ \text{graph 5} \\
+ \frac{s}{2} \text{graph 6} \\
+ \frac{1}{2} \text{graph 7}
\end{array} \]
In two-loop $2 \rightarrow 2$ processes there are two integration momenta and three external momenta $\rightarrow 9$ possible scalar products

Consider the integrals $I_{t,r,s}$ where

- $t \rightarrow \# \text{ of propagators}$
- $9-t \rightarrow \# \text{ of irreducible scalar products}$
- $r \rightarrow \text{sum of the powers of all propagators}$
- $s \rightarrow \text{sum of the powers of all irreducible scalar products}$

The number of integrals belonging to the $I_{t,r,s}$ set is

$$N(I_{t,r,s}) = \binom{r - 1}{r - t} \binom{8 - t + s}{s}$$

It is possible to build $(N_{\text{IBP}} + N_{\text{LI}}) N(I_{t,r,s})$ identities
The identities involve integrals of the class $I_{t,r,s}$ and simpler integrals, but also new unknowns belonging to the classes $I_{t,r+1,s}$ and $I_{t,r+1,s+1}$.

The number of equations grows faster than the number of unknowns.

Number of integrals for $t = 7$

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Number of accumulated equations / unknowns

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**Euler Method**

**First Order Differential Equation**

\[ \frac{d}{dx} f(x) + C(x)f(x) = \Omega(x) \]

1. find the solution of the homogeneous equation

\[ \frac{d}{dx} h(x) + C(x)h(x) = 0 \]

2. build the general solution of the non-homogeneous equation

\[ f(x) = h(x) \left[ k + \int dx \frac{\Omega(x)}{h(x)} \right] \]

3. fix the integration constant \( k \)
Euler Method

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2. build the general solution of the non-homogeneous equation

\[
f(x) = h(x) \left[ k + \int dx \frac{\Omega(x)}{h(x)} \right]
\]

3. fix the integration constant \( k \)
# NNLO Heavy Flavor CS at $\sqrt{s} = 1$ GeV

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$e \ (10^{-4})$</th>
<th>$\mu \ (10^{-4})$</th>
<th>$c \ (10^{-4})$</th>
<th>$\tau \ (10^{-4})$</th>
<th>$b \ (10^{-4})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50°</td>
<td>17.341004</td>
<td>1.7972877</td>
<td>0.0622677</td>
<td>0.0264013</td>
<td>0.0010328</td>
</tr>
<tr>
<td>60°</td>
<td>18.407836</td>
<td>2.2267654</td>
<td>0.0861876</td>
<td>0.0367058</td>
<td>0.0014184</td>
</tr>
<tr>
<td>70°</td>
<td>19.438718</td>
<td>2.6504950</td>
<td>0.1086126</td>
<td>0.0465329</td>
<td>0.0018907</td>
</tr>
<tr>
<td>80°</td>
<td>20.465455</td>
<td>3.0655973</td>
<td>0.1253094</td>
<td>0.0540991</td>
<td>0.0022442</td>
</tr>
<tr>
<td>90°</td>
<td>21.463240</td>
<td>3.4581845</td>
<td>0.1321857</td>
<td>0.0576348</td>
<td>0.0024428</td>
</tr>
<tr>
<td>100°</td>
<td>22.366427</td>
<td>3.8070041</td>
<td>0.1268594</td>
<td>0.0560581</td>
<td>0.0024304</td>
</tr>
<tr>
<td>110°</td>
<td>23.099679</td>
<td>4.0922189</td>
<td>0.1098317</td>
<td>0.0495028</td>
<td>0.0022024</td>
</tr>
<tr>
<td>120°</td>
<td>23.605216</td>
<td>4.3030725</td>
<td>0.0843311</td>
<td>0.0392810</td>
<td>0.0018086</td>
</tr>
<tr>
<td>130°</td>
<td>23.847394</td>
<td>4.4392717</td>
<td>0.0549436</td>
<td>0.0273145</td>
<td>0.0013297</td>
</tr>
</tbody>
</table>

**Table:** The second-order electron, muon, $c$-quark, $\tau$-lepton, and $b$-quark QED contributions to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV in units of $10^{-4}$ of the Born cross section.
NNLO Heavy Flavor CS at $\sqrt{s} = 500$ GeV

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$e \ (10^{-3})$</th>
<th>$\mu \ (10^{-3})$</th>
<th>$\tau \ (10^{-3})$</th>
<th>$t \ (10^{-3})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^\circ$</td>
<td>3.4957072</td>
<td>0.9690710</td>
<td>0.1542329</td>
<td>0.0000575</td>
</tr>
<tr>
<td>$2^\circ$</td>
<td>4.1203687</td>
<td>1.2491270</td>
<td>0.3573661</td>
<td>0.0002466</td>
</tr>
<tr>
<td>$3^\circ$</td>
<td>4.5099086</td>
<td>1.4146106</td>
<td>0.5140242</td>
<td>0.0005763</td>
</tr>
<tr>
<td>$50^\circ$</td>
<td>7.5740980</td>
<td>2.3185800</td>
<td>1.8411736</td>
<td>0.1707137</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>7.7965875</td>
<td>2.3446744</td>
<td>1.9274750</td>
<td>0.2340996</td>
</tr>
<tr>
<td>$70^\circ$</td>
<td>8.0081541</td>
<td>2.3708714</td>
<td>2.0072240</td>
<td>0.2998535</td>
</tr>
<tr>
<td>$80^\circ$</td>
<td>8.2164081</td>
<td>2.3981523</td>
<td>2.0829886</td>
<td>0.3635031</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>8.4172449</td>
<td>2.4207950</td>
<td>2.1521199</td>
<td>0.4202418</td>
</tr>
<tr>
<td>$100^\circ$</td>
<td>8.5982864</td>
<td>2.4282953</td>
<td>2.2085332</td>
<td>0.4655025</td>
</tr>
<tr>
<td>$110^\circ$</td>
<td>8.7451035</td>
<td>2.4090920</td>
<td>2.2456055</td>
<td>0.4979010</td>
</tr>
<tr>
<td>$120^\circ$</td>
<td>8.8465287</td>
<td>2.3536259</td>
<td>2.2585305</td>
<td>0.5181602</td>
</tr>
<tr>
<td>$130^\circ$</td>
<td>8.8954702</td>
<td>2.2543834</td>
<td>2.2446158</td>
<td>0.5287459</td>
</tr>
</tbody>
</table>

**Table:** The second-order electron, $\mu$, $\tau$-lepton, and top-quark contributions to the differential cross section of Bhabha scattering at $\sqrt{s} = 500$ GeV in units of $10^{-3}$ of the Born cross section. The top-quark contribution also includes $\mathcal{O} (\alpha \alpha_s)$. 