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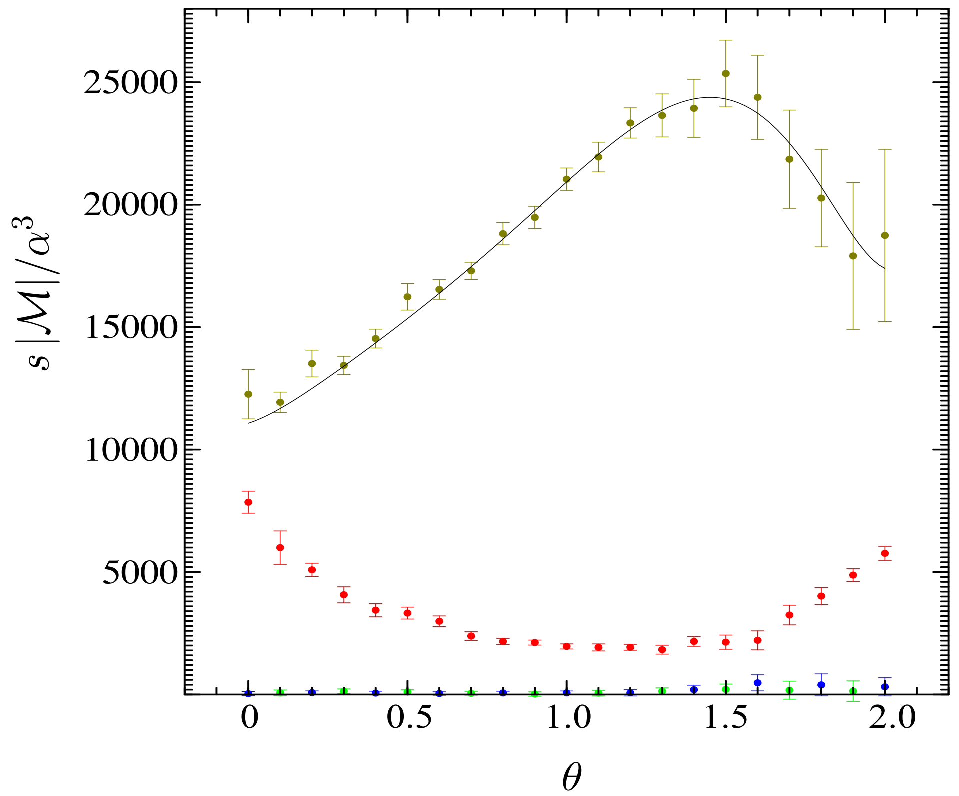
*LoopFest*, FERMILAB, *April 17$^{th}$ 2007*

# Reducing one-loop amplitudes to scalar integrals at the integrand level

- Results
- The method (2-point)
- Generalization to $n$-point amplitudes
- The rational part of the Amplitudes ($n$-point)
- Dealing with numerical inaccuracies (2-point)

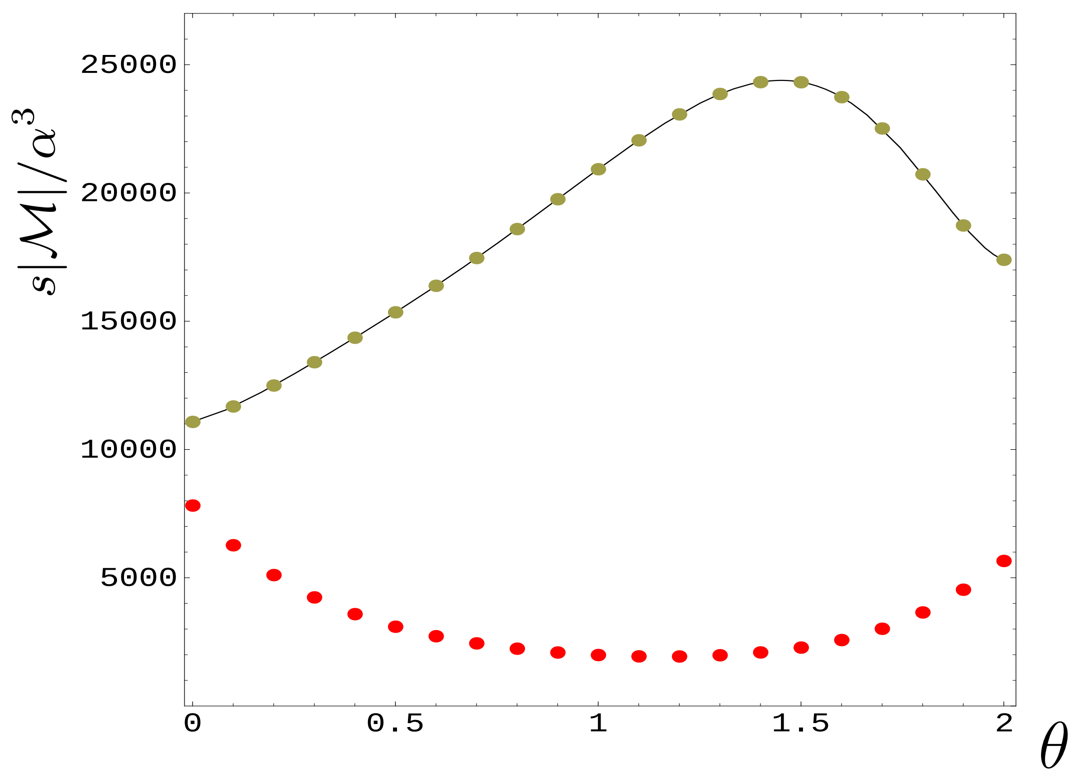
# The benchmark: $2\gamma \to 4\gamma$ with $m_f = 0$




– Z. Nagy and D. E. Soper, Phys. Rev. D **74** (2006) 093006

– T. Binoth, T. Gehrmann, G. Heinrich and P. Mastrolia, hep-ph/0703311

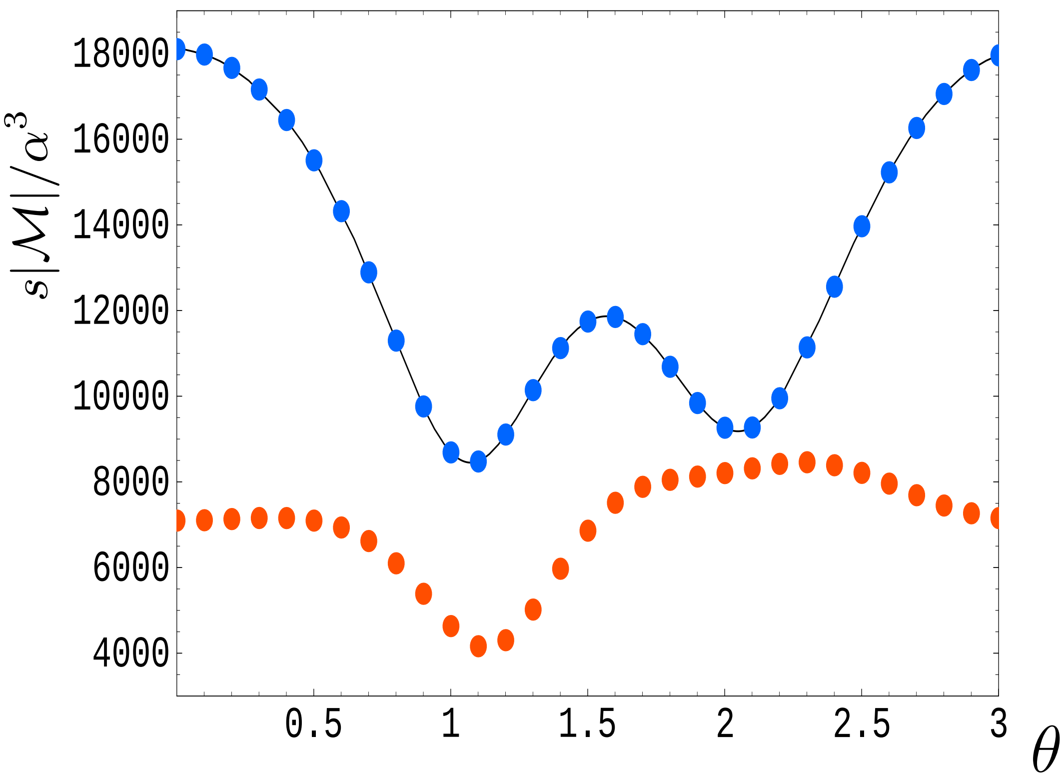
# Our result ($m_f = 0$)



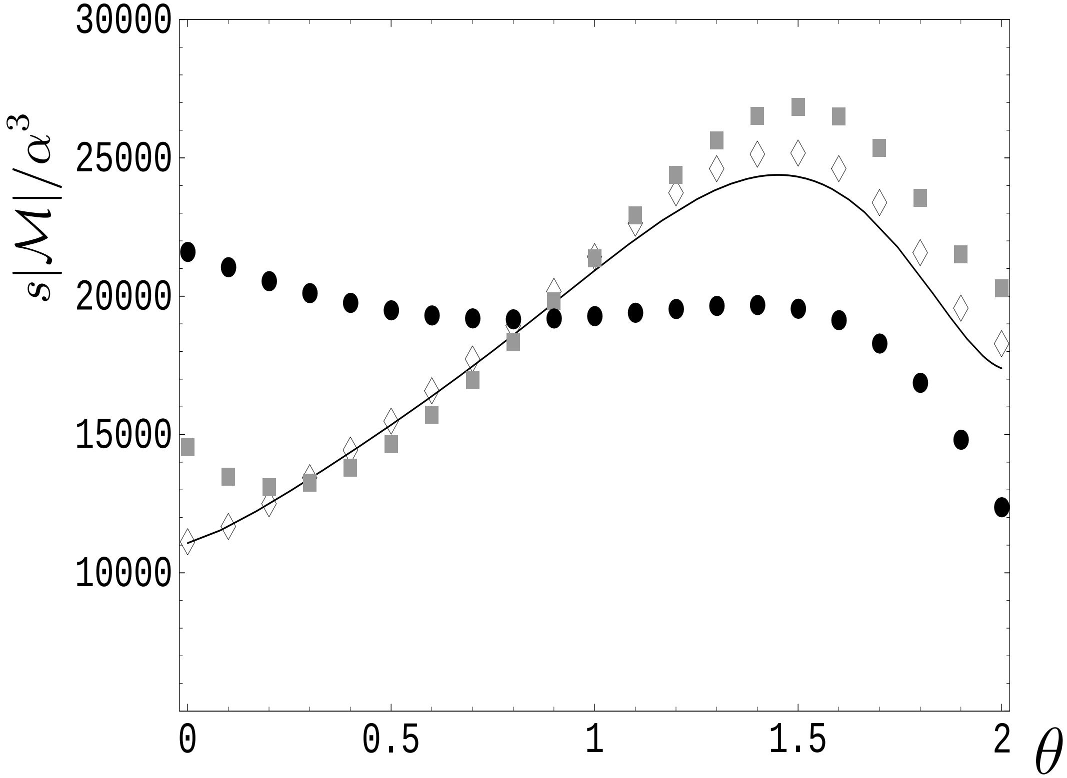

– G. Ossola, C. G. Papadopoulos and R. P. :

- Nucl. Phys. B **763** (2007) 147
- arXiv:0704.1271 [hep-ph]

# A different set of momenta ($m_f = 0$)



# The first set of momenta but $\boxed{m_f \neq 0}$




$m_f = 0.5\ GeV$ (diamond), $m_f = 4.5\ GeV$ (gray box) *and* $m_f = 12\ GeV$ (black dots)

## The Method (2-point)

◇ We start from the *integrand* of a generic 2-point amplitude written in the form

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1}\,, \quad \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2\,, \quad p_0 \neq 0\,,$$

in which we suppose $N(q)$ at most quadratic in $q$

**Rule of the game:**

The analytic form of $N(q)$ SHOULD NOT be used (except information on its maximum rank)

$\Rightarrow$

◇ We expand $N(q)$ as follows:

$$N(q) = [b(01) + \tilde{b}(q;01)] + [a(0) + \tilde{a}(q;0)]D_1 + [a(1) + \tilde{a}(q;1)]D_0$$

*Since we are working at the* integrand *level, we allow for integrands that vanish after integration: the spurious terms $\tilde{b}$ and $\tilde{a}$*

◇ We decompose $q^\mu + p_0^\mu$ in a basis of $k_1 \equiv p_1 - p_0$, $n$ (such that $(n \cdot k_1) = 0$) and 2 more massless 4-vectors $\ell_{7,8}$ (such that $(\ell_{7,8} \cdot n) = (\ell_{7,8} \cdot k_1) = 0$):

$$q^\mu = -p_0^\mu + y_1 k_1^\mu + y_n n^\mu + y_7 \ell_7^\mu + y_8 \ell_8^\mu$$

◇ For example $y_1 = \frac{(q+p_0)\cdot k_1}{k_1^2}$, $y_7 = \frac{(q+p_0)\cdot \ell_8}{\ell_7 \cdot \ell_8}$ $\Rightarrow$

◇ Then $N(q)$ *must* have the form

$$\begin{aligned} N(q) &= b(01) + \tilde{b}_{11}(01)[(q+p_0)\cdot\ell_7] + \tilde{b}_{21}(01)[(q+p_0)\cdot\ell_8] \\ &+ \tilde{b}_{12}(01)[(q+p_0)\cdot\ell_7]^2 + \tilde{b}_{22}(01)[(q+p_0)\cdot\ell_8]^2 \\ &+ \tilde{b}_0(01)[(q+p_0)\cdot n] + \tilde{b}_{00}(01)\,K(q;01) \\ &+ \tilde{b}_{01}(01)[(q+p_0)\cdot\ell_7][(q+p_0)\cdot n] \\ &+ \tilde{b}_{02}(01)[(q+p_0)\cdot\ell_8][(q+p_0)\cdot n] + \mathcal{O}(D_0) + \mathcal{O}(D_1) \end{aligned}$$

with

$$K(q;01) = \left\{ [(q+p_0)\cdot n]^2 - \frac{[(q+p_0)\cdot k_1]^2 - (q+p_0)^2 k_1^2}{3} \right\}$$

◇

Spurious terms naturally appear!!

◇ Notice that, because of the identity

$$2\,(q \cdot k_1) = D_1 - D_0 + (d_1 - d_0)\,, \quad \text{with} \;\; d_i = m_i^2 - p_i^2$$

any term proportional to $[(q + p_0) \cdot k_1]$ either contributes to the constant term $b$ or it is included in the terms $\mathcal{O}(D_{0,1})$ (the same happens for $[(q + p_0) \cdot \ell_7][(q + p_0) \cdot \ell_8]$)

◇

*We suppose to determine the terms $\mathcal{O}(D_{0,1})$*

*at a later stage of the calculation*

◇ Therefore, to determine all the 2-point like coefficients disentangling $\mathcal{O}(D_{0,1})$, we look for a $q$ such that

$$D_0 = D_1 = 0$$

◇ $\Rightarrow$ *Infinite solutions to fit $b$ and all $\tilde{b}$s*

## Generalization to $n$-point amplitudes

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}\,, \qquad \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2 \qquad (p_0 \neq 0)$$

- A bar denotes objects living in $n = 4 + \epsilon$ dimensions:

$$\bar{q}^2 = q^2 + \tilde{q}^2\,,$$

where $\tilde{q}^2$ is $\epsilon$-dimensional, $(\tilde{q} \cdot q) = (\tilde{q} \cdot p_i) = 0$

- Therefore

$$\bar{D}_i = D_i + \tilde{q}^2$$

◇ The numerator $N(q)$ can be cast in the form:

$$\begin{aligned} N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\ &+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\ &+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\ &+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i \\ &+ \tilde{P}(q) \prod_{i}^{m-1} D_i \end{aligned}$$

◇ The quantities $d(i_0 i_1 i_2 i_3)$ are the coefficients of the 4-point loop functions with the four denominators $\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}$. Analogously, the $c(i_0 i_1 i_2)$, $b(i_0 i_1)$ and $a(i_0)$ are the coefficients of all possible 3-point, 2-point and 1-point loop functions, respectively

◇ The "spurious" terms $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$, $\tilde{P}$ still depend on $q$.

*They are defined by the requirement that they should vanish upon integration over* $d^n \bar{q}$

◇ No coefficient of scalar functions with more that four denominators appear: scalar functions with $m > 4$ are expressed in terms of 4-point functions (plus extra $\tilde{d}$'s)

◇ All $q$'s in $N(q)$ are 4-dimensional (if $n$-dimensional $q$'s are needed, they can be split into 4- and $\epsilon$- dimensional parts)

◇ Since the scalar 1-, 2-, 3-, 4-point functions are known, the only knowledge of the existence of such a decomposition allows one to reduce the problem of calculating $A(\bar{q})$ to the

*algebraical problem*

*of extracting all possible coefficients*

*by computing* $N(q)$ *a sufficient number of times*

*at different values of the integration momentum* $q$

*and then inverting the system*

◇ Two problems:

(1) The explicit knowledge of the spurious terms is needed

(2) The system should be kept manageable

($m = 6 \to 56$ scalar one-loop functions + *spurious* terms)

◇ *Assume* (1) *is solved* $\Longrightarrow$ the solution to (2) is singling out particular choices of $q$ such that, systematically, 4, 3, 2 or 1 among all possible denominators $D_i$ vanishes:

*Then the system of equations becomes "triangular": first one solves for all possible 4-point functions, then for the 3-point functions and so on*

◇ *The described procedure can be performed at the* amplitude level $\Rightarrow$ *one does not need to repeat the work for all Feynman diagrams, provided their sum is known*

◇ This is particularly appealing when our method is used together with some recursion relation to build up $N(q)$

◇ The spurious terms can be explicitly constructed as seen in the 2-point case. In general they depend on the maximum rank of $N(q)$

◇ The 4-point case is *special* $\Longrightarrow$

## The $\tilde{d}(q;0123)$ term

◇ There is only one possible integrand that vanishes upon integration

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{k}_1\not{k}_2\not{k}_3\gamma_5]\,, \quad k_i \equiv p_i - p_0$$

$$\Longrightarrow \qquad \tilde{d}(q;0123) = \tilde{d}(0123)\,T(q)$$

where $\tilde{d}(0123)$ is a constant (does not depend on $q$)

◇ This equation is valid for any rank of $N(q)$

$\Longrightarrow$ in any gauge

## The coefficient of the 4-point functions as an example

- ◇ $\tilde{q}^2 = 0 \;\Longrightarrow\; \bar{D}_i \to D_i \;\Longrightarrow$ just missing rational terms

- ◇ We look for a $q$ such that $D_0 = D_1 = D_2 = D_3 = 0$

  $\Longrightarrow$ two possible solutions: $({q_0}^{\pm})^\mu$

- ◇ *At the solutions* one obtains

$$N(q_0^{\pm}) \;=\; [d(0123) + \tilde{d}(0123)\,T(q_0^{\pm})] \prod_{i \neq 0,1,2,3} D_i(q_0^{\pm})$$

◇ Then, by defining

$$R(q_0^\pm) \equiv \frac{N(q_0^\pm)}{\prod_{i\neq 0,1,2,3} D_i(q_0^\pm)}$$

it is possible to extract $d$ and $\tilde{d}$:

$$\begin{aligned} d(0123) &= \frac{1}{2}\left[R(q_0^+) + R(q_0^-)\right], \\ \tilde{d}(0123) &= \frac{1}{2}\frac{R(q_0^+) - R(q_0^-)}{T(q^+)} \end{aligned}$$

◇ The two equations above *do not depend* on the rank of the tensors in the amplitude

◇ When $N(q) = 1$ they allow a trivial decomposition of any $m$-point scalar loop function with $m > 4$ to boxes

◇ The 6-point function simply reads

$$I^6 = \sum_{i<j} \sum_{i,j=0}^{5} d_{ij} I^4(ij)$$

where $I^4(ij)$ is obtained from $I^6$ by dropping the propagators $D_i$ and $D_j$ and where

$$d_{ij} = \frac{1}{2}\left(\frac{1}{D_i(q^+_{(ij)})D_j(q^+_{(ij)})} + \frac{1}{D_i(q^-_{(ij)})D_j(q^-_{(ij)})}\right)$$

## The rational part of the Amplitudes ($n$-point)

◇ One rewrites any denominator in the amplitude as follows

$$\frac{1}{\bar{D}_i} = \frac{\bar{Z}_i}{D_i} \quad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)$$

◇ This results in

$$A(\bar{q}) = \frac{N(q)}{D_0 D_1 \cdots D_{m-1}} \bar{Z}_0 \bar{Z}_1 \cdots \bar{Z}_{m-1}$$

◇ Then, by inserting the expansion of $N(q)$ *in terms of 4-dimensional denominators*, cancellations work exactly and one obtains

$$
\begin{aligned}
A(\bar{q}) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i \; + \; \tilde{P}(q) \prod_{i}^{m-1} \bar{Z}_i
\end{aligned}
$$

*The rational part of the amplitude is then produced, after integrating over* $d^n q$*, by the* $\tilde{q}^2$ *dependence coming from the various* $\bar{Z}_i$

◇ The necessary integrals are

$$I^{(n;2\ell)}_{s;\mu_1\cdots\mu_r} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)} \quad \text{with}$$

$$\bar{D}(k_i) \equiv (\bar{q}+k_i)^2 - m_i^2\,, \quad k_i \equiv p_i - p_0 \quad (k_0 = 0)$$

◇ Such extra-integrals have dimensionality $\mathcal{D} = 2(1+\ell-s)+r$ and give a contribution $O(1)$ only when $\mathcal{D} \geq 0$, otherwise are of $\mathcal{O}(\epsilon)$

◇ *A contribution is developed starting from the 2-point sector* (in renornalizable gauges)

◇ They can be computed *for generic $s$*, namely *for any number of denominators* (5 categories). For example

$$
\begin{aligned}
I^{(n;2(s-1))}_{s;\mu\nu} &= -2i\pi^2 \frac{1}{(s+2)(s+1)s(s-1)} \left\{ \sum_{j=1}^{s} (k_j)_\mu (k_j)_\nu \right. \\
&+ \left. \frac{1}{2} \sum_{j=1}^{s} \sum_{i \neq j}^{s} (k_j)_\mu (k_i)_\nu \right\} + \mathcal{O}(g_{\mu\nu}) + \mathcal{O}(\epsilon)
\end{aligned}
$$

◇ The $g_{\mu\nu}$ part is never needed

◇ The rational part of any $m$-point one-loop amplitude is intimately connected with the form of the *integrand* of the amplitude

◇ Cut-constructible and rational terms are easily obtained, *at the same time*, by solving the same system of linear equations

◇ In the computation of the six-photon amplitude, we *DID NOT* use the additional information on its cut-constructibility and verified only *a-posteriori* that the intermediate rational parts drop out in the final sum

## Dealing with numerical inaccuracies (2-point)

◇ We start from a generic 2-point amplitude

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1}$$

◇ Our purpose is dealing with the situation in which $k_1^2 \equiv (p_1 - p_0)^2 = 0$ *exactly* as well as to set up an algorithm to write down approximations around this case

*with arbitrary precision*

◇ As usual, one expands $N(q)$:

$$\begin{aligned} N(q) &= [b(01) + \tilde{b}(q;01)] \\ &+ [a(0) + \tilde{a}(q;0)] D_1 + [a(1) + \tilde{a}(q;1)] D_0 \end{aligned}$$

◇ If $k_1^2 \to 0$ the described reduction method cannot be applied because the solution to $D_0 = D_1 = 0$, necessary to fit $b(01)$ and $\tilde{b}(q;01)$,

*goes like* $1/k_1^2$ *due to the extra requirement*

$$\boxed{\int d^n q\; \tilde{b}(q;01) = 0}$$

◇ Then, we must consider two separate cases:

($a$) $k_1^2 \to 0$ but $k_1^\mu \neq 0$ (Minkowskian metric)

($b$) $k_1^2 \to 0$ because $k_1^\mu = 0$ (collinear situation)

(a)

- One change basis and still find a solution for which $D_0 = D_1 = 0$

  *by relaxing the further requirement of* $\tilde{b}(q;01)$ *being "spurious"*

- Such a solution goes like $1/(k_1.v)$ ($v$ is an arbitrary massless 4-vector) $\Longrightarrow$ is never singular if $k_1 \neq 0$

- The price to pay is that new *non zero* integrals appear of the type

$$\int d^n q \frac{[(q+p_0)\cdot v]^j}{\bar{D}_0 \bar{D}_1} \quad (j = 1, 2)$$

◇ What has been achieved with this new basis is then moving part of the 1-point functions to the 2-point sector

*in such a way that combinations well behaved in the limit* $k_1^2 \longrightarrow 0$ *appear*

◇ The fact that solutions exist to the condition $D_0 = D_1 = 0$ still allows one to find the coefficients of such integrals (together with all the others)

*without mixing with the 1-point sector*

◇ This solves the problem of reconstructing $N(q)$, but new 2-point integrals must still be computed

◇ Expansions at arbitrary orders in $k_1^2$ can be obtained in terms of loop functions *with less points but higher rank* by observing that

$$\int d^n q \frac{(q\cdot v)^2 (q\cdot k_1)^{2p+1}}{\bar{D}(k_0)\bar{D}(k_1)} = \mathcal{O}(k_1^{2p})$$

$$\int d^n q \frac{(q\cdot v)(q\cdot k_1)^{2p}}{\bar{D}(k_0)\bar{D}(k_1)} = \mathcal{O}(k_1^{2p})$$

and by iteratively using $(f = m_1^2 - k_1^2 - m_0^2)$

$$(q\cdot k_1)^p = \left(\frac{f}{2}\right)^p + \frac{\bar{D}(k_1) - \bar{D}(k_0)}{2} \sum_{i+j=p-1} \left[ (q\cdot k_1)^i \left(\frac{f}{2}\right)^j \right]$$

◇ For example

$$\int d^n q \frac{(q \cdot v)}{\bar{D}(k_0)\bar{D}(k_1)} =$$

$$\frac{1}{f}\int d^n q\,(q \cdot v)\left(\frac{1}{\bar{D}(k_1)} - \frac{1}{\bar{D}(k_0)}\right)\left(1 + \frac{2(q \cdot k_1)}{f}\right) + \mathcal{O}(k_1^2)$$

◇ This procedure breaks down when the quantity $f$ vanishes. In this case a double expansion in $k_1^2$ and $f$ can still be found *in terms of derivatives of one-loop scalar functions*

◇ For example, by multiplying and dividing 2 times by

$$\bar{D}(k_0) = \bar{D}(k_1) - 2(q \cdot k_1) + f$$

one obtains

$$\int d^n q \, \frac{(q \cdot v)}{\bar{D}(k_0)\bar{D}(k_1)} =$$

$$\int d^n q \, \frac{(q \cdot v)}{\bar{D}(k_0)^2} - 2 \int d^n q \, \frac{(q \cdot v)(q \cdot k_1)}{\bar{D}(k_0)^3} + \mathcal{O}(k_1^2) + \mathcal{O}(f)$$

## (b)

◇ In this case *no solution* can be found to the double cut equation $D(k_0) = D(k_1) = 0$ because $D(k_1)$ and $D(k_0)$ are no longer independent:

$$D(k_0) = D(k_1) + f + \mathcal{O}(k_1)$$

◇ One cannot fit separately the coefficients of the 2-point and 1-point functions

◇ We then split the amplitude *from the beginning* by multiplying it by

$$1 \equiv \frac{\bar{D}(k_0) - \bar{D}(k_1)}{f} + \frac{2(q \cdot k_1)}{f}$$

◇ The result is

$$A(\bar{q}) = A^{(1)}(\bar{q}) + A^{(2)}(\bar{q}) + \mathcal{O}(k_1)$$

with

$$A^{(1)}(\bar{q}) = \frac{1}{f}\frac{N(q)}{\bar{D}(k_1)}\,, \quad A^{(2)}(\bar{q}) = -\frac{1}{f}\frac{N(q)}{\bar{D}(k_0)}$$

and $A^{(1,2)}$ can be reconstructed separately, without any problem of vanishing Gram-determinant
(corrections at any order are also calculable)

◇ Once again, when $f \to 0$ *double expansions* in $k_1$ and $f$ can be obtained

◇ For example

$$\begin{aligned} A(q) &= \frac{N(q)}{\bar{D}(k_0)\bar{D}(k_1)} = \frac{N(q)}{\bar{D}(k_0)^2\bar{D}(k_1)}\left[\bar{D}(k_1) - 2(q\cdot k_1) + f\ \right] \\ &= \frac{N(q)}{\bar{D}(k_0)^2} + f\frac{N(q)}{\bar{D}(k_0)^3\bar{D}(k_1)}\left[\bar{D}(k_1) - 2(q\cdot k_1) + f\ \right] + \mathcal{O}(k_1) \\ &= \frac{N(q)}{\bar{D}(k_0)^2} + f\frac{N(q)}{\bar{D}(k_0)^3} + \mathcal{O}(k_1) + \mathcal{O}(f^2) \end{aligned}$$

◇ Similar techniques can also be applied to higher-point amplitudes

# Conclusions

◇ We have shown how computing the *integrand* of any one-loop amplitude *at special values of the integration momentum* allows the *one-shot* reconstruction of all the coefficients of the scalar loop functions *and of the rational terms*

◇ We presented our solution to cure numerical instabilities outlining a strategy to build up expansions at any arbitrary order

◇ We applied our method to the computation of the *massive* six-photon amplitude and we believe it can be successfully used for computing R.C. at LHC and ILC