\( e^+ e^- \rightarrow 3 \) jets at NNLO in QCD

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Motivation

measurements of $e^+e^- \rightarrow \text{jet rates & shape observables}$

- allow precision tests of the Standard Model over wide range of energies
- offer possibility for determination of strong coupling constant $\alpha_s$ with high precision
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- allow precision tests of the Standard Model over wide range of energies
- offer possibility for determination of strong coupling constant $\alpha_s$ with high precision

current error on $\alpha_s$ from jet observables is dominated by theoretical uncertainty:

$$\alpha_s(M_Z) = 0.121 \pm 0.001\text{(exp)} \pm 0.005\text{(theory)} \quad [S.\ Bethke\ 06]$$
determination of $\alpha_s$

- $\alpha_s$ world average:
  determination based only on data where 
  NNLO QCD theory predictions exist!
  (DIS, $\Gamma(Z \to \text{had}), \tau, \Upsilon - \text{decays}$)

- LEP data for jets and shape observables precise, 
  but not used for world average: NNLO theory not available!
determination of $\alpha_s$

- $\alpha_s$ world average:
  - determination based only on data where NNLO QCD theory predictions exist!
    - (DIS, $\Gamma(Z \to \text{had}), \tau, \Upsilon$ decays)
  
- LEP data for jets and shape observables precise, but not used for world average: NNLO theory not available!

- at the LHC: $n$-jet cross section $\sim \alpha_s^n$
  - $\Rightarrow$ reducing error on $\alpha_s$ very important
determination of $\alpha_s$

$\alpha_s$ world average:

determination based only on data where NNLO QCD theory predictions exist!

(DIS, $\Gamma(Z \rightarrow \text{had})$, $\tau$, $\Upsilon$ – decays)

LEP data for jets and shape observables precise, but not used for world average: NNLO theory not available!

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$\Rightarrow$ reducing error on $\alpha_s$ very important

future International Linear Collider: will reach precision at the per mille level
**event shapes**

**event shape observables:**

- characterize global properties of hadronic events
- easily accessible experimentally

**example thrust:**

\[ T = \max_n \left( \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|} \right) \]

two-jet limit ("pencil-like event"): \( T \to 1 \)

"spherical" event: \( T \to 1/2 \)

poorly convergent perturbative expansion for \( T \to 1 \)

\[ \Rightarrow \text{resummation, power corrections} \]

\[ e^+e^- \to 3 \text{ jets at NNLO in QCD} \quad \text{p.4} \]
jet modelling

inclusion of higher orders in perturbation theory

- reduction of scale dependence
- allow to model jet structure in a realistic way and to predict absolute rates

\[ e^+ e^- \rightarrow 3 \text{ jets at NNLO in QCD} \]

LO: one jet = one parton
NLO: up to two partons form one jet
NNLO: up to three partons form one jet
**NLO calculations**

**virtual:** explicit infrared poles from loop integrals

\[ d\sigma^V = \frac{P_2}{\epsilon^2} + \frac{P_1}{\epsilon} + P_0 \]

**real:** infrared poles from phase space regions where partons become soft and/or collinear

⇒ need **subtraction terms** \( d\sigma^S \)

⇒ \[ \int_{\text{sing}} d\sigma^S = -\frac{P_2}{\epsilon^2} - \frac{P_1}{\epsilon} + Q_0 \]

\[ \sigma^{NLO}_{m\ jet} = \int_{m+1} \left[ d\sigma^R - d\sigma^S \right]_{\epsilon=0} + \int_m \left[ \begin{array}{c} \text{analytically} \quad \int d\sigma^V \\ \text{analytically} \quad \int_1 d\sigma^S \end{array} \right]_{\epsilon=0} \]

Numerically

\( e^+e^- \rightarrow 3 \text{ jets at NNLO in QCD} \)
NNLO calculations

problems in NNLO calculations:

- massless particles $\Rightarrow$ IR singularities entangled in a complicated way
- enormous complexity of expressions
- analytic integrations very difficult
- direct numerical evaluation hampered by singularities

IR poles need to be isolated and subtracted

- from loop integrals
  ($\rightarrow$ integrations up to two loops $\rightarrow 1/\epsilon$ poles explicit)
- from soft/collinear phase space regions
  ($\rightarrow$ subtraction terms)
m-jet production schematically:

\[ d\sigma_{NNLO} = \int d\Phi_{m+2} \left( d\sigma^R_{NNLO} - d\sigma^S_{NNLO} \right) + \int d\Phi_{m+2} d\sigma^S_{NNLO} \]

\[ + \int d\Phi_{m+1} \left( d\sigma^{V,1}_{NNLO} - d\sigma^{VS,1}_{NNLO} \right) + \int d\Phi_{m+1} d\sigma^{VS,1}_{NNLO} \]

\[ + \int d\Phi_m d\sigma^{V,2}_{NNLO} \]

\[ \int d\Phi_{X'''} d\sigma^S_{NNLO} + \int d\Phi_{X'} d\sigma^{VS,1}_{NNLO} + d\sigma^{V,2}_{NNLO} = \text{finite} \]
subtraction of infrared poles: NNLO

two conceptually different approaches:

- "conventional" approach: (generalization of NLO procedure)
  manual construction of a subtraction scheme and analytic integration over subtraction terms in $D = 4 - 2\epsilon$ dimensions
  [A. Gehrmann-De Ridder, T. Gehrmann, N. Glover, GH], [Del Duca, Somogyi, Trocsanyi], [Catani, Frixione, Grazzini], [Kilgore], [Weinzierl]

- sector decomposition: automated isolation of IR poles in parameter space and numerical integration of pole coefficients
  [Binoth, GH], [Anastasiou, Melnikov, Petriello]
partonic ingredients for $e^+e^- \rightarrow 3$ jets at NNLO

- 2-loop virtual
  3-parton phase space

- one-loop plus up to one final state parton unresolved
  4-parton phase space

- purely real emission up to 2 partons soft and/or collinear
  5 parton phase space
antenna subtraction method

NLO: D. Kosower '98; J. Campbell, M. Cullen, N. Glover '98
NNLO: D. Kosower '02,03; A. Gehrmann-DeRidder, T. Gehrmann, N. Glover '05,06

based on colour ordered amplitudes
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$e^+ e^- \rightarrow 3$ jets at NNLO in QCD – p.11
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- based on **colour ordered** amplitudes
- allows **symmetric** treatment of emitter/spectator
- antenna functions can be at **tree** level or at **one loop**
- radiators are a colour ordered pair of hard partons
  - hard quark-antiquark pair
  - hard quark-gluon pair
  - hard gluon-gluon pair
- can be derived from physical matrix elements:
  [A.Gehrmann-De Ridder, T.Gehrmann, N.Glover ’05]
  \[ q\bar{q} \text{ from } \gamma^* \rightarrow q\bar{q} + X, \quad qg \text{ from } \chi \rightarrow g\bar{g} + X, \quad gg \text{ from } H \rightarrow gg + X \]

\[ e^+e^- \rightarrow 3 \text{ jets at NNLO in QCD – p.11} \]
antenna subtraction

unresolved configurations:

double unresolved:  
• triple collinear  
• double single collinear  
• double soft  
• soft/collinear

single unresolved:  
• collinear  
• soft

single unresolved → three-parton antenna $X^0_3(i, j, k)$
double unresolved → four-parton antenna $X^0_4(i, j, k, l)$
### types of antenna functions

<table>
<thead>
<tr>
<th>Tree Level</th>
<th>One Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>$qg\bar{q}$</td>
<td>$A_3^0(q, g, \bar{q})$, $\tilde{A}_3^1(q, g, \bar{q})$, $\hat{A}_3^1(q, g, \bar{q})$</td>
</tr>
<tr>
<td>$qgq\bar{q}$</td>
<td>$A_4^0(q, g, g, \bar{q})$, $\tilde{A}_4^0(q, g, g, \bar{q})$</td>
</tr>
<tr>
<td>$qq'\bar{q}'\bar{q}$</td>
<td>$B_4^0(q, q', q', \bar{q})$, $\tilde{B}_4^0(q, q', q', \bar{q})$, $\hat{B}_4^0(q, q', q', \bar{q})$</td>
</tr>
<tr>
<td>$qq\bar{q}q$</td>
<td>$C_4^0(q, q, q, \bar{q})$, $\tilde{C}_4^0(q, q, q, \bar{q})$, $\hat{C}_4^0(q, q, q, \bar{q})$</td>
</tr>
<tr>
<td>$qgg$</td>
<td>$D_3^0(q, g, g)$, $\hat{D}_3^1(q, g, g)$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$qq'\bar{q}'$</td>
<td>$E_3^0(q, q', q')$, $\tilde{E}_3^1(q, q', q')$, $\hat{E}_3^1(q, q', q')$</td>
</tr>
<tr>
<td>$qq'\bar{q}'g$</td>
<td>$E_4^0(q, q', q', g)$, $\tilde{E}_4^0(q, q', q', g)$, $\hat{E}_4^0(q, q', q', g)$</td>
</tr>
<tr>
<td>$ggg$</td>
<td>$F_3^0(g, g, g)$, $\hat{F}_3^1(g, g, g)$</td>
</tr>
<tr>
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<td>$F_4^0(g, g, g, g)$, $\hat{F}_4^1(g, g, g, g)$</td>
</tr>
<tr>
<td>$gg\bar{q}$</td>
<td>$G_3^0(g, q, \bar{q})$, $\tilde{G}_3^1(g, q, \bar{q})$, $\hat{G}_3^1(g, q, \bar{q})$</td>
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<td>$G_4^0(g, q, q, \bar{q})$, $\tilde{G}_4^0(g, q, q, \bar{q})$, $\hat{G}_4^0(g, q, q, \bar{q})$</td>
</tr>
<tr>
<td>$q\bar{q}'q'$</td>
<td>$H_4^0(q, q, q', q')$, $\tilde{H}_4^0(q, q, q', q')$, $\hat{H}_4^0(q, q, q', q')$</td>
</tr>
</tbody>
</table>
Examples

Single unresolved:

\[ \tilde{13}, \tilde{23} \text{ linear combinations of } p_1, p_2, p_3 \text{ ("momentum mappings") } [D. Kosower '02] \]

\[ |M_{qgqg\bar{q}}(1, 3, 4, 5, 2)|^2 \xrightarrow{3 \text{ unres.}} |M_{qgqg\bar{q}}(\tilde{13}, 4, 5, \tilde{23})|^2 \times A_3^0(1, 3, 2) \]

\[ A_3^0(1, 3, 2) \sim |M_{qg\bar{q}}|^2 / |M_{q\bar{q}}|^2 \]

Antenna function \( A_3^0(1, 3, 2) \):

describes all configurations where gluon 3 is unresolved between \( q\bar{q} \)-pair

\[ e^+ e^- \rightarrow 3 \text{ jets at NNLO in QCD -- p.14} \]
single unresolved

phase space factorisation:

\[ d\phi_{m+1}(p_1, \ldots, p_i, p_j, p_k, \ldots, p_{m+1}) \]
\[ \rightarrow \quad d\phi_m(p_1, \ldots, \tilde{p}_I, \tilde{p}_K, \ldots, p_{m+1}) \times d\Phi X_{ijk} \]

\( \tilde{p}_I, \tilde{p}_K \): results of momentum mappings

\( \mathcal{F}^{(3\rightarrow 2)} : \{p_i, p_j, p_k\} \rightarrow \{\tilde{p}_I, \tilde{p}_K\} \)

subtraction terms integrated analytically over antenna PS:

\[ X_{ijk}^0 = \int d\Phi X_{ijk} X_3^0(i, j, k) \]

combine with virtual corrections to cancel poles

\( e^+ e^- \rightarrow 3 \) jets at NNLO in QCD – p.15
\[ d\sigma_{NNLO}^R - d\sigma_{NNLO}^S = \mathcal{N} \sum_{m+2} d\Phi_{m+2}(p_1, \ldots, p_{m+2}) \frac{1}{S_{m+2}} \]

\[ \times \left[ |\mathcal{M}_{m+2}(p_1, \ldots, p_{m+2})|^2 J^{(m+2)}_m(p_1, \ldots, p_{m+2}) \right. \]

\[ - \sum X^0_3(i, j, k) |\mathcal{M}_{m+1}(p_1, \ldots, \tilde{p}_I, \tilde{p}_K, \ldots, p_{m+2})|^2 \]

\[ \left. \times J^{(m+1)}_m(p_1, \ldots, \tilde{p}_I, \tilde{p}_K, \ldots, p_{m+2}) \right] \]

\( J^{(n)}_m \): measurement function constructing \( m \) jets from \( n \) partons
double unresolved

\[ \frac{\Delta M_{\gamma g g q^*}(1, 3, 4, 5, 2)}{\Delta M_{\gamma g q^*}(134, 5, 234)} \sim |M_{\gamma g g q^*}|^2 / |M_{q g q^*}|^2 \]

4-parton antenna functions \( X_4^0(i, j, k, l) \):

can become singular in single unresolved limits which do not match limits of ME

\( \Rightarrow \) these limits need to be subtracted with appropriate PS mapping
\[
\begin{align*}
\frac{d\sigma_{NNLO}^R - d\sigma_{NNLO}^S}{N} &= \sum_{m+2} \frac{d\Phi_{m+2}(p_1, \ldots, p_{m+2})}{S_{m+2}} \\
&\times \left[ |\mathcal{M}_{m+2}(p_1, \ldots, p_{m+2})|^2 J_{m}^{(m+2)}(p_1, \ldots, p_{m+2}) \\
&- \sum X^0_4(i, j, k, l) |\mathcal{M}_m(p_1, \ldots, \tilde{p}_I, \tilde{p}_L, \ldots, p_{m+2})|^2 \\
&\times J_{m}^{(m)}(p_1, \ldots, \tilde{p}_I, \tilde{p}_L, \ldots, p_{m+2}) \right]
\end{align*}
\]
momentum maps

momentum mappings have to satisfy:

- momentum conservation, i.e. 
\[ \tilde{p}_I + \tilde{p}_L = p_i + p_j + p_k + p_l \]
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- \( \tilde{p}_I \) and \( \tilde{p}_L \) should reduce to appropriate original momenta in exact singular limits
- the mapping should not introduce spurious singularities
- mappings follow the ones suggested by D. Kosower \[\text{[hep-ph/0212097]}\]
colour structure at NNLO

\[
\sigma_{\text{NNLO}} = (N^2 - 1) \left[ N^2 \sigma_{N^2} + \sigma_{N^0} + \frac{1}{N^2} \sigma_{1/N^2} \right] + N_F N \sigma_{N_F N} + \frac{N_F}{N} \sigma_{N_F/N} + N_F^2 \sigma_{N_F^2} + N_{F,\gamma} \left( \frac{4}{N} - N \right) \sigma_{N_{F,\gamma}}
\]

\[
N_{F,\gamma} = \frac{(\sum_q e_q)^2}{\sum_q e_q^2}
\]

\(N_{F,\gamma}\) contribution numerically negligible

each colour structure evaluated separately
Numerical implementation

starting point:

\[ \text{EERAD2 : } e^+ e^- \rightarrow 4j \]  
(based on NLO antenna subtraction)

[Campbell, Cullen, Glover ’98]

\[ e^+ e^- \rightarrow 3 \text{ jets at NNLO in QCD } \]
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uniform mappings of momenta in subtraction terms
[D. Kosower ’02]
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phase space: decomposed into wedges defined by relative sizes of invariants
(allow optimal generation of phase space points in unresolved limits)

type (a): $(s_{ij}, s_{jk})$ smallest

type (b): $(s_{ij}, s_{kl})$ smallest
(e.g. 5-parton PS: 30 type (a) wedges, 15 type (b) wedges)

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combines PS points related by a rotation of system of unresolved partons
\[ \Rightarrow \text{angular correlations} \] largely cancel by summing over permutations

\[ e^+e^- \rightarrow 3 \text{ jets at NNLO in QCD – p.21} \]
Numerical implementation

Monte Carlo Phase Space

Cross section

Jet Algorithm

Histograms

$\frac{d\sigma}{dT}^{S,1} - \frac{d\sigma}{dT}^{S,0} \rightarrow 3$ jet

$\frac{d\sigma}{dS}^{NLO} \rightarrow 3$ jet

$\frac{d\sigma}{dC}^{NLO} \rightarrow 3$ jet

$e^+e^- \rightarrow 3$ jets at NNLO in QCD – p.22
subtraction of infrared poles: NNLO

recall \( m \)-jet production schematically:

\[
d\sigma_{NNLO} = \int d\Phi_{m+2} \left( d\sigma_{NNLO}^R - d\sigma_{NNLO}^S \right) + \int d\Phi_{m+2} d\sigma_{NNLO}^S \\
+ \int d\Phi_{m+1} \left( d\sigma_{NNLO}^{V,1} - d\sigma_{NNLO}^{VS,1} \right) + \int d\Phi_{m+1} d\sigma_{NNLO}^{VS,1} \\
+ \int d\Phi_m d\sigma_{NNLO}^{V,2} \\

\int d\Phi_m \left\{ \int d\Phi_{X_4} d\sigma_{NNLO}^S + \int d\Phi_{X_3} d\sigma_{NNLO}^{VS,1} + d\sigma_{NNLO}^{V,2} \right\} = \text{finite}
\]

\( e^+ e^- \rightarrow 3 \) jets at NNLO in QCD – p.23
Numerical behaviour

computation time and numerical error dominated by 5-parton channel
(4p channel takes less than 10% of the time)

$e^+ e^- \rightarrow 3$ jets at NNLO in QCD – p.24
Numerical behaviour

- computation time and numerical error dominated by 5-parton channel
  (4p channel takes less than 10% of the time)

- speed: between 6h \( \left( \frac{N_F}{N} \right) \) and 36h \( N^0 \) for 120 \( \times \) 20M points (in 5p channel)
  (2.8 GHz Athlon processors)
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- checks:
  - independence of lower cut $y_0$ on phase space variables ($y_0 \sim 10^{-7} - 10^{-5}$)
  - local cancellations along phase space trajectories approaching singular limits checked using RAMBO
    [Kleiss, Stirling, S.D.Ellis]
  - distributions in genuine phase space variables

$e^+e^- \rightarrow 3$ jets at NNLO in QCD – p.24
thrust distribution

\[ (1 - T') \frac{1}{\sigma_0} \frac{d\sigma}{dT} = \left( \frac{\alpha_s}{2\pi} \right) A(T) \]

\[ + \left( \frac{\alpha_s}{2\pi} \right)^2 \left\{ B(T) - 2A(T) \right\} \]

\[ + \left( \frac{\alpha_s}{2\pi} \right)^3 \left\{ C(T) - 2B(T) + 1.64 A(T) \right\} \]

\[ \frac{\sigma^{\text{had}}}{\sigma^0} = 1 + \frac{3}{2} C_F \frac{\alpha_s}{2\pi} + \sum_{n \geq 2} K_n(\mu^2) \left( \frac{\alpha_s}{2\pi} \right)^n \]

cut: \( (1 - T)_{\text{min}} = 0.01 \)
Results

leading colour $N^2$

colour factor $N^0$
Results

colour factor \( \frac{1}{N^2} \)

colour factor \( N_F N \)
Results

colour factor $N_F/N$

colour factor $N_F^2$

$e^+ e^- \rightarrow 3 \text{ jets at NNLO in QCD – p.28}$
Summary

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$e^+ e^- \rightarrow 3$ jets at NNLO in QCD – p.29
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extension to hadronic initial states
→ talk of Daniel Maître
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- extension to hadronic initial states
  → talk of Daniel Maître
- we are looking forward to the ILC!
backup slides

e^+ e^- \rightarrow 3 \text{ jets at NNLO in QCD – p.30}
momentum maps for single unresolved

\[ \tilde{p}_I = x p_i + r p_j + z p_k \]
\[ \tilde{p}_K = (1 - x) p_i + (1 - r) p_j + (1 - z) p_k \]

\[ x = \frac{1}{2(s_{ij} + s_{ik})} \left[ (1 + \rho) s_{ijk} - 2 r s_{jk} \right] \]
\[ z = \frac{1}{2(s_{jk} + s_{ik})} \left[ (1 - \rho) s_{ijk} - 2 r s_{ij} \right] \]

\[ \rho^2 = 1 + \frac{4 r (1 - r) s_{ij} s_{jk}}{s_{ijk} s_{ik}} \]

\( r \) can be chosen conveniently, here \( r = \frac{s_{jk}}{s_{ij} + s_{jk}} \)

[D. Kosower ’02]
momentum maps for double unresolved

\[
\tilde{p}_{j_1} = x p_i^1 + r_1 p_i^2 + r_2 p_i^3 + z p_i^4 \\
\tilde{p}_{j_2} = (1 - x) p_i^1 + (1 - r_1) p_i^2 + (1 - r_2) p_i^3 + (1 - z) p_i^4 \\
r_1 = \frac{s_{23} + s_{24}}{s_{12} + s_{23} + s_{24}}, \quad r_2 = \frac{s_{34}}{s_{13} + s_{23} + s_{34}} \\
x = \frac{1}{2(s_{12} + s_{13} + s_{14})} \left[ (1 + \rho) s_{1234} - r_1 (s_{23} + 2 s_{24}) - r_2 (s_{23} + 2 s_{34}) \\
+ (r_1 - r_2) \frac{s_{12 s_{34}} - s_{13 s_{24}}}{s_{14}} \right] \\
z = \frac{1}{2(s_{14} + s_{24} + s_{34})} \left[ (1 - \rho) s_{1234} - r_1 (s_{23} + 2 s_{12}) - r_2 (s_{23} + 2 s_{13}) \\
- (r_1 - r_2) \frac{s_{12 s_{34}} - s_{13 s_{24}}}{s_{14}} \right] \\
\rho = \left[ 1 + \frac{(r_1 - r_2)^2}{s_{14}^2 s_{1234}^2} \lambda(s_{12} s_{34}, s_{14} s_{23}, s_{13} s_{24}) \right. \\
+ \frac{1}{s_{14} s_{1234}} \left\{ 2 r_1 s_{12} [(1 - r_2) s_{34} + 2 (1 - r_1) s_{24}] \\
+ 2 r_2 s_{13} [(1 - r_1) s_{24} + 2 (1 - r_2) s_{34}] \\
- 2 s_{14} s_{23} [r_1 (1 - r_2) + r_2 (1 - r_1)] \right\}^{\frac{1}{2}} \]

$e^+ e^- \rightarrow 3$ jets at NNLO in QCD – p.32